University of Alberta

MULTI-LOOP CONTROLLER SYNTHESIS AND PERFORMANCE ANALYSIS

by

Vinay Kariwala ©

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Canadä

I simply picked a bunch of flowers
and added nothing
but the thread that binds them

-Michel De Montaigne, French writer (1533-1592)

To my family
&
Fraser Forbes, my Ph.D. thesis supervisor

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Nomenclature

The frequently used symbols in this report are included in the following list. The vectors are written in lower case bold and matrices in upper case bold. The individual elements of a matrix are written in lower case of the same symbol as used for the matrix.

Main Notation

$\dot{(\cdot)}$	Time derivative
$(\cdot)_i$	i^{th} element of vector, i^{th} column of matrix
$(\cdot)_i^{'}$	i^{th} row of matrix
$(\cdot)_{ij}$	ij^{th} element of matrix
	Sub-matrix made of rows and columns indexed by sets i and j
$(\cdot)^{ij}$	Matrix with i^{th} row and j^{th} column deleted
$\|\cdot\|_p$	p-norm of vector, matrix or transfer matrix
$(\cdot)^T$	Transpose
$(\cdot)^*$	Complex conjugate transpose
$(\cdot)^{-*}$	Complex conjugate transpose of the inverse
$(\cdot)\circ(\cdot)$	Hadamard or element-wise product
$(\cdot)\succ(\cdot)$	Partial ordering, $\mathbf{A}\succ\mathbf{0}$ implies $\mathbf{A}$ is positive definite
$\mathrm{Re}(\cdot)$	Real part
$\mathrm{Im}(\cdot)$	Imaginary part
$\det(\cdot)$	Determinant
$\mathrm{tr}(\cdot)$	Trace
$\mathrm{diag}(\cdot)$	Matrix formed by direct matrix sum of the elements (blocks)
$\mathrm{E}[\cdot]$	Expectation operator
$(\cdot)!$	Factorial, $n! = \prod_{i=1}^{n} i$
$(\cdot)\bigcup(.)$	Union of sets
$(\cdot)\bigcap(\cdot)$	Intersection of sets
$\leftrightarrow$	Minimal state space realization of transfer matrix
j	Imaginary number, $\sqrt{-1}$
$m_i \times m_j$	Dimension of the i^{th} diagonal block of the partitioned system
n_y	Number of outputs of a transfer matrix
n_u	Number of inputs of a transfer matrix
n_z	Number of zeros of a transfer matrix
n_p	Number of poles of a transfer matrix
p	Pole of the transfer matrix

z	Zero of the transfer matrix
s	Laplace variable
q^{-1}	Back shift operator
I_2	Achievable $\mathcal{H}_2$ optimal input performance
I_∞	Achievable $\mathcal{H}_\infty$ optimal input performance
$\mathbf{1}_n$	n-dimensional vector of ones
$\mathbf{u}$	Manipulated variables, inputs
$\mathbf{y}$	Controlled variables, outputs
$\mathbf{w}$	Disturbance variables, Exogenous inputs
$\mathbf{u}_p$	Input pole direction associated with pole p
$\mathbf{u}_z$	Input zero direction associated with zero z
$\mathbf{y}_p$	Output pole direction associated with pole p
$\mathbf{y}_z$	Output zero direction associated with zero z
$\mathbf{A}$	State matrix in the linear state-space realization
$\mathbf{B}$	Input matrix in the linear state-space realization
$\mathbf{C}$	Output matrix in the linear state-space realization
$\mathbf{D}$	Matrix with the direct effect of $\mathbf{u}$ on $\mathbf{y}$ in the linear state-space realization, Scaling matrix, Interactor matrix
$\mathbf{P}$	Diagonal state matrix with poles as diagonal elements in the state-space realization
$\mathbf{F}$	State feedback gain
$\mathbf{L}$	Observer gain
$\mathbf{T}$	State transformation matrix
$\mathbf{I}$	Identity matrix
$\mathbf{X}$	Solution of state feedback algebraic Riccati equation
$\mathbf{Y}$	Solution of observer algebraic Riccati equation
$\mathbf{C}(s)$	Compensator
$\mathbf{G}(s)$	Transfer matrix connecting controlled and manipulated variables
$\mathbf{G}_{mi}(s)$	Input minimum phase part of $\mathbf{G}(s)$
$\mathbf{G}_{mo}(s)$	Output minimum phase part of $\mathbf{G}(s)$
$\mathbf{G}_{si}(s)$	Input stable part of $\mathbf{G}(s)$
$\mathbf{G}_{so}(s)$	Output stable part of $\mathbf{G}(s)$
$\mathbf{G}_w(s)$	Transfer matrix connecting controlled and disturbance variables
$\mathcal{U}(\mathbf{G}(s))$	Unstable part of $\mathbf{G}(s)$
$\mathbf{K}(s)$	Controller
$\mathcal{B}_{zi}(s)$	Blaschke product obtained by input factorization of RHP zeros
$\mathcal{B}_{zo}(s)$	Blaschke product obtained by output factorization of RHP zeros
$\mathcal{B}_{pi}(s)$	Blaschke product obtained by input factorization of RHP poles
$\mathcal{B}_{po}(s)$	Blaschke product obtained by output factorization of RHP poles
$\mathbf{S}(s)$	Sensitivity function
$\mathbf{T}(s)$	Complementary sensitivity function
$\mathbf{T}_{zw}(s)$	Closed loop transfer matrix from $\mathbf{z}$ to $\mathbf{w}$
$\mathbf{W}_u(s)$	Frequency dependent weight for input performance

$\mathcal{RH}_\infty$	Subspace of rational stable transfer matrices with real coefficients
$\mathbb{R}^{m \times n}$	$m \times n$ dimensional space of real numbers
$\mathbb{C}^{m \times n}$	$m \times n$ dimensional space of complex numbers
$N(\alpha, (.))$	Number of clockwise encirclements of $(\alpha, 0)$ by image of Nyquist D contour under (.)

Greek Symbols

η	Performance Index
κ	Euclidian condition number of matrix
μ	Structured singular value
$\bar{\mu}$	Upper bound on structured singular value obtained by scaling
ρ	Spectral radius
λ	Eigenvalue
$\underline{\lambda}$	Minimum eigenvalue
σ	Singular value
$\bar{\sigma}$	Maximum singular value
$\underline{\sigma}$	Minimum singular value
σ_H	Hankel singular value (see Definition 2.4)
$\bar{\sigma}_H$	Maximum Hankel singular value (see Definition 2.4)
$\underline{\sigma}_H$	Minimum Hankel singular value (see Definition 2.4)
ω	frequency
λ_{ij}	Relative gain between $\mathbf{y}_i$ and $\mathbf{u}_j$
$\mathbf{\Lambda}$	Relative gain array
$[\mathbf{\Lambda}_B]_{ij}$	Block relative gain between $\mathbf{y}_i$ and $\mathbf{u}_j$
θ	Time delay for a SISO transfer matrix
$\mathbf{\Theta}$	Time delay for a MIMO transfer matrix
$\mathbf{\Delta}$	Uncertainty, perturbation matrix
$\mathbf{\Gamma}$	Performance Relative Gain Array

Abbreviations

iff	if and only if
wrt	with respect to
ARE	Algebraic Riccati equation
BRG	Block relative gain
CCD	Control configuration design
CSD	Control structure design
FIR	Finite impulse response
GBDD	Generalized block diagonal dominance
LTI	Linear time invariant
LHP	Left half of complex plane
LHS	Left hand side
MIMO	Multi input Multi output

MV	Minimum variance
PID	Proportional integral derivative
PRGA	Performance relative gain array
QBDD	Quasi block diagonal dominance
RHP	Right half of complex plane
RHS	Right hand side
RGA	Relative gain array
SISO	Single input single output

Norms

Induced 2-norm: For a $m \times n$ matrix, $\mathbf{A}$,

$$\|\mathbf{A}\|_2 = \sup_{\|\mathbf{u}\|_2=1} \|\mathbf{A}\mathbf{u}\|_2 = \bar{\sigma}(\mathbf{A})$$

$\mathcal{H}_2$ norm: For a stable and strictly proper transfer matrix $\mathbf{G}(s)$,

$$\|\mathbf{G}(s)\|_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\left(\mathbf{G}(j\omega)^*\mathbf{G}(j\omega)\right) d\omega$$

$\mathcal{H}_\infty$ norm: For a stable transfer matrix $\mathbf{G}(s)$,

$$\|\mathbf{G}(s)\|_\infty = \sup_{\mathrm{Re}(s)>0} \bar{\sigma}(\mathbf{G}(s)) = \sup_{\omega\in\mathbb{R}} \bar{\sigma}(\mathbf{G}(j\omega))$$

$\mathcal{L}_\infty$ norm: Similar to $\mathcal{H}_\infty$ norm, except that $\mathbf{G}(s)$ can be unstable.

Hankel norm: For a stable transfer matrix $\mathbf{G}(s)$,

$$\|\mathbf{G}(s)\|_H = \bar{\sigma}_H(\mathbf{G}(s))$$

Chapter 1

Introduction

1.1 The Case for Decentralized Control

For a multivariate system, it is mathematically attractive to use a centralized controller to meet the desired objectives of stabilization and performance requirements. In practice, a set of smaller dimensional controllers, which make their decisions locally, is frequently used. A control strategy that uses a set of non-interacting controllers is called a decentralized control strategy. Formally defining [102],

> ***Decentralized controller*** *is a control system consisting of non-interacting feedback controllers, which interconnect a set of output measurements/commands with a subset of manipulated inputs. These subsets should not be used by any other controller.*

In general, a centralized controller provides better performance and constraint handling as compared to the decentralized controllers. On the other hand, in addition to their inherent simplicity, a decentralized control system exhibit several advantages over a fully centralized design. In the ideal case, these advantages include [18, 102]:

1. The individual controller subsystems can be brought in and out of service providing flexibility of operation in presence of changing control objectives.
2. Due to the localized effect of the individual controllers, the system can be made fault tolerant with ease, particularly in the case of a sensor or actuator failure.
3. The individual controllers are easier to tune online in presence of changing process conditions.
4. Simpler models can be used to design and tune the controllers reducing the modelling requirements.

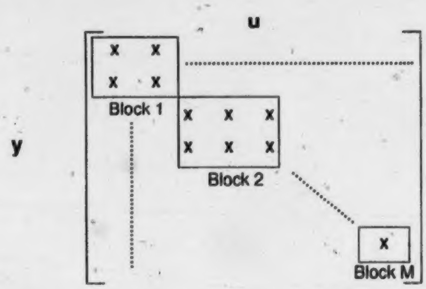

Figure 1.1: Block-wise system partitioning

5. The online computational effort is less than their multivariable counterparts and implementation is simpler.

For a given system, all of the mentioned advantages may not be realized simultaneously or may only be realized at the cost of degraded performance. Nevertheless, decentralized control seems to be the almost exclusive choice for control of large-scale systems.

For power systems, decentralized control is necessitated due to physical distances between different stations and the enormous cost of establishing a communication network. In process systems, the use of decentralized controllers is motivated by the difficulty (and impossibility) of obtaining reliable dynamic models and ease of tuning and design. Decentralized control is sometimes implicit in non-conventional systems such as the administrative system of a country, where the provincial governments look after the welfare of citizens under the supervision of federal government. Decentralized control is also the preferred choice by nature, *e.g.* the secretion of different enzymes and hormones in the human body is controlled by different sections of the brain.

1.2 Motivation and Scope

Before a decentralized control scheme can be implemented, suitable pairings between the controlled and the manipulated variables need to be determined. In other words, the system needs to be partitioned into a number of blocks (see Figure 1.1). In some cases such as a platoon of vehicles, the partitioning can be obvious. In the general case, there exist competing alternatives for partitioning and the choice depends on the design requirements.

Consider the example of an industrial boiler furnace [94], where the objective is to control the temperatures (y) by manipulating the gas flow rates (u) in the four boilers.

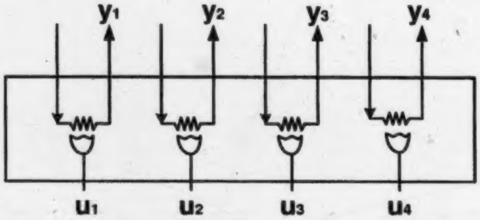

Figure 1.2: Industrial boiler furnace

For this system, the $\mathbf{y}_1, \mathbf{y}_2$ are primarily affected by $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{y}_3, \mathbf{y}_4$ by $\mathbf{u}_3, \mathbf{u}_4$. When the system is partitioned as $((\mathbf{y}_1 - \mathbf{y}_2, \mathbf{u}_1 - \mathbf{u}_2), (\mathbf{y}_3 - \mathbf{y}_4, \mathbf{u}_3 - \mathbf{u}_4))$, a block decentralized controller can be designed easily to closely match the closed loop performance of the centralized controller [83]. If the objective is to instead obtain acceptable closed loop performance with minimum controller complexity, a fully decentralized controller with $((\mathbf{y}_1, \mathbf{u}_1), (\mathbf{y}_2, \mathbf{u}_2), (\mathbf{y}_3, \mathbf{u}_3), (\mathbf{y}_4, \mathbf{u}_4))$ partitioning suffices.

The problem of pairing controlled and manipulated variables, or system partitioning is known as control configuration design (CCD) problem. This thesis aims at developing tools for solving the CCD problem. At this point, it is fair to question the necessity of seeking a systematic solution to the CCD problem. After all, decentralized controllers, designed based on heuristics and process knowledge, have been successfully used in large-scale process industries for decades.

Due to the increased competitiveness and tighter environmental regulations, the levels of mass, energy and information integration among process units have increased drastically over the years. The controllers designed optimally for every unit do not always work well together. Luyben *et al.* [79] report that process control lore contains tales of multi-million dollar plants, that never operated. Thus, the work in this thesis is primarily motivated by the increased complexity of the systems.

The second reason is pure intellectual curiosity and the drive to make things better. The heuristics used for partitioning process systems and subsequently designing control systems are a result of the invaluable experience acquired by the process engineers over the years through trial and error. A sound mathematical theory for solving the CCD problem can provide valuable insight into the advantages and possibly unknown disadvantages of these heuristics closing the gap between theory and practice [36]. Simultaneously, these insights can be used for meeting the desired objectives closely with reduced controller complexity [86].

The CCD problem itself is a sub-problem of the more general control structure design (CSD) problem. In the CSD problem, the tasks of identifying controlled and manipulated variables from measurements, determining pairings between them and selecting the controller type are dealt with simultaneously or sequentially [102, 107].

Throughout this thesis, we assume that the sets of controlled and manipulated variables have already been identified. For process systems, the set of manipulated variables is easily selected as the valve inputs that can be varied independently, but the choice of controlled variables is not always obvious. Recently, Skogestad [99] proposed the promising method of self-optimizing control for selection of controlled variables based on economics. Govatsmark [46] has demonstrated the usefulness of this approach through industrial-scale case studies. A review of some other methods available for the selection of the sets of controlled and manipulated variables is available in [107].

Some other assumptions and conventions used in this thesis are in order. It is assumed that the system can be described by a finite dimensional linear time invariant (LTI) model, which is available. Considering the difficulty associated with procuring a reliable dynamic model, parts of this thesis focus on using simple models such as steady state gain model, as far as possible. With slight abuse of notation, the following terms are used interchangeably: system and FDLTI model, controlled variables and outputs and, manipulated variables and inputs. A block diagonal matrix is generally perceived as a matrix with the block sub-matrices being square. In this thesis, the same term is used, when the individual blocks are possibly non-square. When the inverse of a matrix or a system is used, it is assumed that it exists. For simplicity, the same symbol is used for inverse of square and left or right inverse of non-square matrices and systems. To emphasize the structure of the controller, the decentralized controller is referred to as the fully decentralized controller for the diagonal controller and block decentralized controller otherwise.

1.3 Thesis Overview

During the past two decades, the CCD or the pairing problem has drawn a lot of attention from researchers, particularly in the area of process control. An overview of the available methods can be found in [102] and a more detailed review in [106]. With the variety of methods available, this thesis aims at addressing some of the relevant issues that have received little attention. Whereas some of the results are extensions and generalizations of the available results, some new concepts are also introduced. This thesis can be broadly divided into three parts:

1. System stabilization using multivariate or decentralized controller (Chapters 2 and 3)
2. Pairing selection for the stabilized system (Chapters 4 and 5)
3. Performance monitoring of decentralized controllers (Chapter 6)

An overview of the individual chapters of the thesis follows.

System stabilization Most (if not all) pairing selection tools are developed under the assumption that the underlying system is stable. In Chapter 2, we characterize the achievable input performance of linear systems possibly having time delay operating under feedback control. Based on these results, a simple iterative method is presented for selection of a subset of controlled and manipulated variables for pre-stabilizing the system using a multivariate controller.

In Chapter 3, we propose a methodology for synthesizing the stabilizing decentralized controller using independent designs. The methodology involves a paradigm shift, as the decentralized controller is designed based on a block diagonal approximation of the system instead of the block diagonal elements. A numerical solution for finding the optimal block diagonal approximation through minimization of scaled $\mathcal{L}_\infty$ distance between the system and the approximation is presented.

Pairing selection Contrary to the SISO pairings, block pairings are still selected based on heuristics [19, 29]. For systematic selection of block pairings, we study a promising method, *i.e.* block relative gain (BRG) [83] in Chapter 4. The connections between BRG and issues like closed loop stability, controllability, block diagonal dominance and interactions are explored and simple pairing rules are proposed. As an offshoot, we develop a number of algebraic properties of BRG.

In Chapter 5, we show that the recently proposed necessary and sufficient conditions [52] for assessing integrity of a system, can be equivalently expressed in terms of well known notions of BRG and Niedrilinski's index [49, 87]. These results imply that establishing the existence of a diagonal controller with integral action such that the system has integrity is NP-hard [41].

Performance monitoring The responsibilities of a control engineer extend well beyond ensuring good performance at design stage. Sustained benefits can result from monitoring the control system performance and proper maintenance when performance degrades. In Chapter 6, we point out the insufficiency of the available minimum variance (MV) benchmark [69] for performance monitoring of decentralized controllers. We present an approximate solution to the decentralized MV benchmark problem, where the upper bound on the output variance is minimized. Though a similar numerical search based method has recently been available [114], the suboptimal solution presented here is explicit and is also extended for performance monitoring of multi-loop PID controllers.

For the readers convenience, an overview of the relevant concepts from the linear systems, control and optimization theory is presented in every chapter. Advanced readers can skip these portions of the thesis without loss of continuity.

Chapter 2

Input Performance Limitations of Feedback Control

For selecting controlled and manipulated variables to stabilize the system, we characterize the achievable input performance for linear time invariant (LTI) systems with and without time delay. Achievable input performance depends primarily on the joint controllability and observability of unstable poles in both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control frameworks. A simple method is presented for the extended stability problem, where unstable as well as stable poles close to the imaginary axis of complex plane are moved to a half complex plane. We draw a number of insights that are useful for selection of variables for stabilizing layer, as well as process design and formulation of the optimal controller design problem. [1]

2.1 Introduction

For complex unstable systems, often the requirements of stabilization and performance satisfaction are separated, *i.e.* a subset of controlled and manipulated variables is initially used for stabilization and then another controller is designed for the stabilized system to satisfy the performance requirements. The question remains: Which controlled and manipulated variables should be used for stabilization? These variables can be conveniently selected such that the input or control effort required for stabilization is minimized as [58]:

---

[1] This work was performed while the author was visiting Professor Sigurd Skogestad, Norwegian Institute of Science and Technology, Trondheim, Norway during March-May 2003.

Parts of this chapter were presented at the annual meeting of American Institute of Chemical Engineers, San Francisco, CA, 2003 and the American Control Conference, Boston, MA, 2004 [74].

(i) the likelihood of input saturation is reduced;

(ii) the disturbing effect of the stabilizing control layer on the stabilized system is minimized; and

(iii) generally output performance is not very important for stabilizing control.

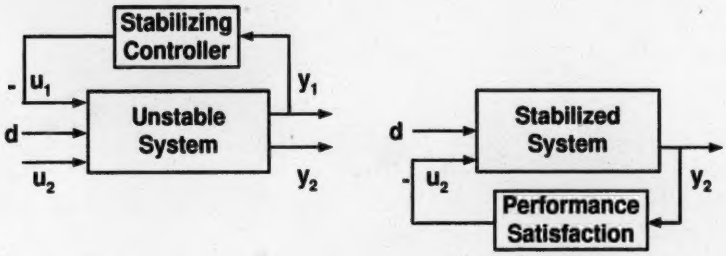

Figure 2.1: Separation of controller design objectives

In Figure 2.1, let the set of controlled variables, $\mathbf{y}$ and manipulated variables, $\mathbf{u}$ be partitioned as, $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2]$ and $\mathbf{u} = [\mathbf{u}_1\ \mathbf{u}_2]$. The variables for the stabilizing layer $(\mathbf{y}_1, \mathbf{u}_1)$ are selected such that the closed loop system is stable and the norm of the transfer matrix from disturbances $\mathbf{d}$ to $\mathbf{u}_1$ is minimized. For this purpose, we characterize the achievable input performance of LTI systems under feedback control in this chapter. Then, the variables of the stabilizing layer can be selected by simply comparing the input requirement for stabilization using different subsets of variables. It is pointed out, however, that for any meaningful comparison, it is necessary to scale the variables of system prior to analysis. The possible choices for scaling factors include: maximum allowable ranges [102] or variance and the economic penalty associated with variation of individual variables.

In the $\mathcal{H}_2$ control framework, the problem of control effort minimization is the dual of the well studied minimum variance or cheap control problem [69, 92]. It is known that the output performance of the system is limited by its unstable zeros and time delay. Similarly, the unstable poles and time delays pose limitations on the achievable input performance. In the context of stable systems, some authors [64, 80, 102] have considered characterizing the achievable input performance for disturbance rejection under the assumption of perfect control. The focus of this chapter is on stabilization and note that the minimal control effort required for stabilizing stable system is trivially zero.

The broad area of fundamental performance limitations has drawn a lot of interest in the past two decades. An overview of the available results and some recent developments in this area can be found in [27, 97, 102] and the references within. Though the focus has largely been on obtaining bounds on sensitivity and complementary sensitivity functions, which primarily address output performance issues [22], some researchers have considered characterizing achievable input performance directly or indirectly.

Glover [43] studied the robust stability of systems in the presence of additive unstructured uncertainty. With this description of uncertainty, maximizing robust stability is equivalent to minimizing the $\mathcal{H}_\infty$ norm of transfer matrix from disturbances to inputs. Clearly, these results are relevant to the problem in the present context, but the disturbance model and frequency dependent weight are assumed to be minimum phase stable. Havre and Skogestad [57] relaxed this assumption of minimum phase stable disturbance model and frequency dependent weight and derived expressions for the lower bound on achievable input performance. Using a novel approach of pole vectors, the same authors [58] have provided exact expressions for rational systems with single unstable pole driven by measurement noise. Chen *et. al.* [26] have studied the optimal regulation problem with input usage penalized for rational unstable systems driven by input disturbances in the $\mathcal{H}_2$ optimal control framework. These results can be related to the present problem by appropriate choice of weights.

In this chapter, we characterize the minimal input requirement for stabilization in both of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control frameworks. The system is considered to be driven by output disturbances, where the disturbance model can share unstable poles with the system. This representation poses no limitations and the case of input disturbances is easily handled by setting the disturbance model same as the system. We further generalize these results to systems with input-output time delay. In addition to selection of variables for stabilization, the results presented here are also useful in process design considering achievable control performance and optimal controller synthesis problem formulation.

For a specified set of controlled and manipulated variables, the control effort required for stabilization can be easily calculated using available numerical techniques for optimal controller design. In addition to the computational expense involved, a limitation of such a numerical approach is that it does not provide any information regarding the factors limiting the input performance. These insights are useful for making appropriate design modifications, when the system cannot be stabilized by constraining the inputs of the system within their maximal allowable ranges. In some special cases, these insights can also provide simple analytic methods for selection of variables for stabilizing layer [58].

The organization of the remaining discussion in this chapter is as follows: key results from linear systems theory including optimal control are reviewed in § 2.2; the problem of designing the optimal controller that minimizes input usage for stabilization is formulated and simplified in § 2.3; the achievable input performance for univariate and multivariate systems is characterized in § 2.4 and § 2.5, respectively; in § 2.6, we present a simple method for the extended stability problem, where unstable as well as stable poles close to the imaginary axis are moved to a half complex plane; we present some insights and an iterative algorithm to reduce the computational complexity involved in selecting controlled and manipulated variables for stabilizing control in § 2.7; and § 2.8 concludes this chapter.

2.2 Preliminaries

In this section, we collect some general results from linear systems theory. These results form the basis for further development in this chapter.

2.2.1 Poles and Zeros

The notions of poles and zeros for univariate systems are generally well understood. For multivariate systems, the poles and zeros are characterized by their locations as well as directions. As a consequence, contrary to univariate systems, a multivariate system can have poles and zeros at the same location with no cancellation if the associated directions are different. The knowledge of pole and zero directions provides a simple method for factorization of systems into an all-pass factor and a minimum phase or stable part, as discussed later. We briefly review the concepts of poles and zeros of multivariate systems, where the discussion is adapted from [56, 102].

For a univariate system, z_i is a zero of $g(s)$ if $g(z_i) = 0$. This definition of zeros can be generalized to multivariate systems by noting that at $s = z_i$, the rank of $g(s)$ reduces from 1 to 0.

Definition 2.1 $z_i \in \mathbb{C}$, $i = 1, 2 \cdots n_z$ are called the zeros of $\mathbf{G}(s)$ if the rank of $\mathbf{G}(z_i)$ is less than the normal rank of $\mathbf{G}(s)$. The normal rank of $\mathbf{G}(s)$ is $\mathbf{G}(s)$ evaluated at all $s \notin \{z_i\}$ [81].

Based on the above definition, it follows that z_i are the zeros of $\mathbf{G}(s)$ iff there exists non-zero $\mathbf{u}_{z_i}, \mathbf{y}_{z_i}$ such that

$$\begin{array}{rrcllr} & \mathbf{G}(z_i)\mathbf{u}_{z_i} & = & \mathbf{0} & \text{and} & \mathbf{G}(s)\mathbf{u}_{z_i} \neq \mathbf{0} \quad \forall s \neq z_i \\ \text{and} & \mathbf{y}_{z_i}^* \mathbf{G}(z_i) & = & \mathbf{0} & \text{and} & \mathbf{y}_{z_i}^* \mathbf{G}(s) \neq \mathbf{0} \quad \forall s \neq z_i \end{array}$$

where $\mathbf{u}_{z_i}, \mathbf{y}_{z_i}$ are usually normalized to have unit length. The $\mathbf{u}_{z_i}$ and $\mathbf{y}_{z_i}$ are called the input and output zero directions respectively corresponding to the zero z_i. The zeros of the multivariate system are sometimes called transmission zeros, but are simply referred as zeros in this thesis. Let the quadruplet $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be a minimal state space realization of $\mathbf{G}(s)$ represented as $\mathbf{G}(s) \leftrightarrow (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$. The zeros and the associated zero directions of $\mathbf{G}(s)$ are easily determined by solving the following generalized eigenvalue problems:

$$\begin{bmatrix} \mathbf{A} - z_i\mathbf{I} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{z_i} \\ \mathbf{u}_{z_i} \end{bmatrix} = \mathbf{0}; \qquad \begin{bmatrix} \mathbf{v}_{z_i}^* & \mathbf{y}_{z_i}^* \end{bmatrix} \begin{bmatrix} \mathbf{A} - z_i\mathbf{I} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \mathbf{0}$$

Definition 2.2 $p_i \in \mathbb{C}, i = 1, 2 \cdots n_p$ are called poles of $\mathbf{G}(s)$ if one or more elements of $\mathbf{G}(s)$ fails to be analytic (becomes infinite) in the complex plane [8].

With a slight abuse of terminology, the poles of $\mathbf{G}(s)$ can be alternatively defined as the zeros of $\mathbf{G}^{-1}(s)$. Then it follows that p_i are the poles of $\mathbf{G}(s)$ iff there exists non-zero $\mathbf{u}_{p_i}, \mathbf{y}_{p_i}$ such that

$$\begin{array}{rrcllr} & \mathbf{u}_{p_i}^*\mathbf{G}^{-1}(p_i) & = & \mathbf{0} & \text{and} & \mathbf{u}_{p_i}^*\mathbf{G}^{-1}(s) \neq \mathbf{0} \quad \forall s \neq p_i \\ \text{and} & \mathbf{G}^{-1}(p_i)\mathbf{y}_{p_i} & = & \mathbf{0} & \text{and} & \mathbf{G}^{-1}(s)\mathbf{y}_{p_i} \neq \mathbf{0} \quad \forall s \neq p_i \end{array}$$

where $\mathbf{u}_{p_i}, \mathbf{y}_{p_i}$ are usually normalized to have unit length. The $\mathbf{u}_{p_i}$ and $\mathbf{y}_{p_i}$ are called the input and output pole directions respectively corresponding to the pole p_i. For a system with distinct poles, let $\mathbf{G}(s) \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{P}$ is a diagonal matrix. Then it can be shown that

$$\mathbf{u}_{p_i}^T = \mathbf{B}_i'/\|\mathbf{B}_i'\|_2; \qquad \mathbf{y}_{p_i} = \mathbf{C}_i/\|\mathbf{C}_i\|_2$$

where $\mathbf{B}_i'$ and $\mathbf{C}_i$ denote the i^{th} row and i^{th} column of $\mathbf{B}$ and $\mathbf{C}$ respectively. When the system has repeated poles, the expressions for calculating input and output pole directions are more complex and are available in [56].

2.2.2 All Pass Factorization of RHP Poles and Zeros

Definition 2.3 A square transfer matrix $\mathbf{G}(s)$ is called all-pass (also called square paraconjugate unitary rational matrix) if $\mathbf{G}(j\omega)\mathbf{G}^*(-j\omega) = \mathbf{I}$ for all $\omega \in \mathbb{R}$.

A linear system with RHP poles and zeros can be factored into an all-pass factor and a minimum phase or stable part. Such a factorization is useful for manipulation and simplification of expressions arising later in this chapter. The two popular approaches for all-pass factorization of linear systems are inner-outer factorization and the use of Blaschke products. For univariate systems, both these approaches produce identical results. For multivariate systems, use of Blaschke products provides analytical expressions and is preferred over inner-outer factorization in which solution of algebraic Riccati equations (AREs) is required. The idea of using Blaschke products for factorization of RHP poles and zeros was introduced by Wall *et al.* [111] and was used for characterization of achievable performance by Chen [21, 22] and Havre [57]. A collection of some of the useful properties of Blaschke products is available in [56].

Let $z_i \in \mathbb{C}, i = 1 \cdots n_z$ be the non-minimum phase or RHP zeros of $\mathbf{G}(s)$. Then $\mathbf{G}(s)$ can be factored as follows:

$$\mathbf{G}(s) = \mathbf{G}^1(s)\mathcal{B}_1(s) \qquad \mathcal{B}_1(s) = \mathbf{I} - \frac{2\mathrm{Re}(z_1)}{s+\bar{z}_1}\hat{\mathbf{u}}_{z_1}\hat{\mathbf{u}}_{z_1}^* \tag{2.1}$$

where $\hat{\mathbf{u}}_{z_1}$ is the input zero direction of z_1. With this factorization, z_1 is not a zero of $\mathbf{G}^1(s)$. By repeated application of (2.1) on $\mathbf{G}^i(s)$, $i = 1 \cdots n_z - 1$, $\mathbf{G}(s)$ can be factored into a minimum-phase part and an all pass filter as,

$$\mathbf{G}(s) = \mathbf{G}_{mi}(s)\mathcal{B}_{zi}(s) \qquad \mathcal{B}_{zi}(s) = \prod_{i=1}^{n_z}\left(\mathbf{I} - \frac{2\mathrm{Re}(z_i)}{s+\bar{z}_i}\hat{\mathbf{u}}_{z_i}\hat{\mathbf{u}}_{z_i}^*\right) \tag{2.2}$$

In (2.2), $\mathbf{G}_{mi}(s)$ is minimum phase with the RHP zeros of $\mathbf{G}_m(s)$ mirrored across the imaginary axis and $\mathcal{B}_{zi}(s)$ is an all pass filter. Note that except the direction associated with the zero factored first, $\hat{\mathbf{u}}_{z_i}$ differs from $\mathbf{u}_{z_i}$, as it is calculated based on $\mathbf{G}^{(i-1)}(s)$. The RHP zeros can be alternatively factored at system's output as follows:

$$\mathbf{G}(s) = \mathcal{B}_{zo}(s)\mathbf{G}_{mo}(s) \qquad \mathcal{B}_{zo}(s) = \prod_{i=n_z}^{1}\left(\mathbf{I} - \frac{2\mathrm{Re}(z_i)}{s+\bar{z}_i}\hat{\mathbf{y}}_{z_i}\hat{\mathbf{y}}_{z_i}^*\right) \tag{2.3}$$

When $\mathbf{G}(s)$ has RHP poles at $p_i \in \mathbb{C}, i = 1 \cdots n_p$, these poles can also be factored into a stable part and an all pass filter on the input and output side as follows:

$$\mathbf{G}(s) = \mathbf{G}_{si}(s)\mathcal{B}_{pi}^{-1}(s) \qquad \mathcal{B}_{pi}^{-1}(s) = \prod_{i=n_p}^{1}\left(\mathbf{I} - \frac{2\mathrm{Re}(p_i)}{s-p_i}\hat{\mathbf{u}}_{p_i}\hat{\mathbf{u}}_{p_i}^*\right) \tag{2.4}$$

$$\mathbf{G}(s) = \mathcal{B}_{po}^{-1}(s)\mathbf{G}_{so}(s) \qquad \mathcal{B}_{po}^{-1}(s) = \prod_{i=1}^{n_p}\left(\mathbf{I} - \frac{2\mathrm{Re}(p_i)}{s-p_i}\hat{\mathbf{y}}_{p_i}\hat{\mathbf{y}}_{p_i}^*\right) \tag{2.5}$$

2.2.3 Optimal Control

In this chapter, we use a state-space approach for characterization of achievable input performance. For this purpose, we briefly review the pioneering results on $\mathcal{H}_2$ and $\mathcal{H}_\infty$

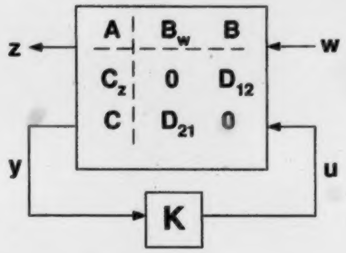

Figure 2.2: Generalized plant for optimal controller design

optimal control due to Doyle *et al.* [34]. Further details can be found in many recently published textbooks dealing with optimal control (*e.g.* [47, 117]). In later sections, we show how these general results simplify when input performance is maximized.

With reference to Figure 2.2, let $\mathbf{z}$ and $\mathbf{w}$ denote the exogenous outputs and inputs and, $\mathbf{y}$ and $\mathbf{u}$ be the measured and manipulated variables respectively. The model of the generalized plant from $\mathbf{w}$ to $\mathbf{z}$ has the following form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_w\mathbf{w} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}_{21}\mathbf{w} \\ \mathbf{z} &= \mathbf{C}_z\mathbf{x} + \mathbf{D}_{12}\mathbf{u} \end{aligned} \tag{2.6}$$

Assumption 2.1 System (2.6) is assumed to be in the standard form [34]:

(a) $(\mathbf{A}, \mathbf{B}_w)$ is stabilizable and $(\mathbf{A}, \mathbf{C}_z)$ is detectable.

(b) $(\mathbf{A}, \mathbf{B})$ is stabilizable and $(\mathbf{A}, \mathbf{C})$ is detectable.

(c) $\mathbf{D}_{12}^*\mathbf{D}_{12} = \mathbf{I}$ and $\mathbf{D}_{21}^*\mathbf{D}_{21} = \mathbf{I}$.

(d) $\mathbf{D}_{12}^*\mathbf{C}_z = \mathbf{0}$ and $\mathbf{D}_{21}^*\mathbf{B}_w = \mathbf{0}$.

In addition, the assumptions that $\mathbf{D}_{11} = 0$ and $\mathbf{D}_{22} = 0$ are implicit in the realization of the generalized plant (2.6). The assumption that $\mathbf{D}_{22} = 0$ can be easily satisfied by a linear fractional transformation on the controller $\mathbf{K}(s)$ [117, pp. 261]. $\mathbf{D}_{11} = 0$ is necessary for well-posedness of the the $\mathcal{H}_2$ optimal control problem. In general, this assumption can be relaxed for the $\mathcal{H}_\infty$ optimal control problem, but this complicates the formulae substantially. Some additional details on the physical interpretation of Assumption 2.1 and transforming the problem to satisfy them can be found in [102, p. 363].

It follows from Assumption 2.1(a)-(b) that there exist $\mathbf{X}_2, \mathbf{Y}_2 \succeq 0$, which solve the following algebraic Riccati equations (AREs),

$$\begin{aligned} \mathbf{A}^*\mathbf{X}_2 + \mathbf{X}_2\mathbf{A} - \mathbf{X}_2\mathbf{B}\mathbf{B}^*\mathbf{X}_2 + \mathbf{C}_z^*\mathbf{C}_z &= 0 \\ \mathbf{A}\mathbf{Y}_2 + \mathbf{Y}_2\mathbf{A}^* - \mathbf{Y}_2\mathbf{C}^*\mathbf{C}\mathbf{Y}_2 + \mathbf{B}_w\mathbf{B}_w^* &= 0 \end{aligned}$$

Let $\mathbf{T}_{zw}$ be the closed loop transfer matrix from w to z. The unique controller minimizing $\|\mathbf{T}_{zw}(s)\|_2$ is given as [34]:

$$\mathbf{K}_{\text{opt}}(s) = \left[\begin{array}{c|c} \mathbf{A} + \mathbf{B}\mathbf{F}_2 + \mathbf{L}_2\mathbf{C} & -\mathbf{L}_2 \\ \hline \mathbf{F}_2 & 0 \end{array}\right] \tag{2.7}$$

where $\mathbf{F}_2 = -\mathbf{B}^*\mathbf{X}_2$, $\mathbf{L}_2 = -\mathbf{Y}_2\mathbf{C}^*$ and the optimal cost is [117],

$$I_2^2 = \inf_{\mathbf{K}(s)} \|\mathbf{T}_{zw}(s)\|_2^2 = \text{tr}(\mathbf{B}_w^*\mathbf{X}_2\mathbf{B}_w) + \text{tr}(\mathbf{F}_2\mathbf{Y}_2\mathbf{F}_2^*) \tag{2.8}$$

For the minimization of $\|\mathbf{T}_{zw}(s)\|_\infty$, let $\mathbf{X}_\infty, \mathbf{Y}_\infty \succeq 0$ solve the following algebraic Riccati equations,

$$\mathbf{A}^*\mathbf{X}_\infty + \mathbf{X}_\infty\mathbf{A} - \mathbf{X}_\infty(\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^* - \mathbf{B}\mathbf{B}^*)\mathbf{X}_\infty + \mathbf{C}_z^*\mathbf{C}_z = 0 \tag{2.9}$$

$$\mathbf{A}\mathbf{Y}_\infty + \mathbf{Y}_\infty\mathbf{A}^* - \mathbf{Y}_\infty(\gamma^{-2}\mathbf{C}_z^*\mathbf{C}_z - \mathbf{C}^*\mathbf{C})\mathbf{Y}_\infty + \mathbf{B}_w\mathbf{B}_w^* = 0 \tag{2.10}$$

where $\gamma > 0$. The existence of $\mathbf{X}_\infty, \mathbf{Y}_\infty \succeq 0$ that solve the AREs (2.9)- (2.10) is guaranteed, if Assumption 2.1 holds and $\rho(\mathbf{X}_\infty\mathbf{Y}_\infty) < \gamma^2$. A suboptimal controller achieving $\|\mathbf{T}_{zw}(s)\|_\infty < \gamma$ is [34]:

$$\mathbf{K}_{\text{sub}}(s) = \left[\begin{array}{c|c} \mathbf{A} + \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^*\mathbf{X}_\infty + \mathbf{B}\mathbf{F}_\infty + \mathbf{Z}_\infty\mathbf{L}_\infty\mathbf{C} & -\mathbf{Z}_\infty\mathbf{L}_\infty \\ \hline \mathbf{F}_\infty & 0 \end{array}\right] \tag{2.11}$$

where $\mathbf{F}_\infty = -\mathbf{B}^*\mathbf{X}_\infty$, $\mathbf{L}_\infty = -\mathbf{Y}_\infty\mathbf{C}^*$ and $\mathbf{Z}_\infty = (\mathbf{I} - \gamma^{-2}\rho(\mathbf{X}_\infty\mathbf{Y}_\infty))^{-1}$. The optimal cost is given as

$$I_\infty = \inf_{\mathbf{K}(s)} \|\mathbf{T}_{zw}(s)\|_\infty = \rho^{\frac{1}{2}}(\mathbf{X}_\infty\mathbf{Y}_\infty) \tag{2.12}$$

2.2.4 Hankel Singular Values and Balanced Realization

It is shown later in this chapter that the achievable input performance of a system primarily depends on the Hankel singular values of the image of the unstable part of the system. The concept of Hankel singular values is introduced next.

Definition 2.4 For a rational stable system $\mathbf{G}(s) \leftrightarrow (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, let $\mathbf{X}_H, \mathbf{Y}_H \succeq 0$ solve the following Lyapunov equations,

$$\begin{aligned} \mathbf{A}\mathbf{X}_H + \mathbf{X}_H\mathbf{A}^* + \mathbf{B}\mathbf{B}^* &= 0 && (2.13) \\ \mathbf{A}^*\mathbf{Y}_H + \mathbf{Y}_H\mathbf{A} + \mathbf{C}^*\mathbf{C} &= 0 && (2.14) \end{aligned}$$

Then, the Hankel singular values of $\mathbf{G}(s)$, $\sigma_{Hi}(\mathbf{G}(s))$ are given as $\sigma_{Hi}(\mathbf{G}(s)) = \lambda_i^{1/2}(\mathbf{X}_H\mathbf{Y}_H)$ [42, 117].

Note that the Hankel singular values are independent of the $\mathbf{D}$ matrix of the state space realization of the system. This follows as the $\mathbf{D}$ matrix represents the direct effect of inputs on outputs, but the Hankel singular values measure the effect of past inputs on future outputs [42].

The matrices $\mathbf{X}_H$ and $\mathbf{Y}_H$ are called the controllability and observability gramians of the system. If all the poles of the system are controllable $\mathbf{X}_H \succ 0$. In this sense, the larger the eigenvalues of $\mathbf{X}_H$ are, the more controllable are the modes of the system. Similar conclusions can be drawn for the observability of modes based on the eigenvalues of $\mathbf{Y}_H$. As $\sigma_{Hi}(\mathbf{G}(s)) = \lambda_i^{1/2}(\mathbf{X}_H\mathbf{Y}_H)$, the Hankel singular values are often referred to as the measure of the joint controllability and observability of the modes of the system.

It is well known that the state space realization of a system is not unique. Let $\mathbf{T}$ be a non-singular state transformation matrix. Then, if $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is one realization of the system $\mathbf{G}(s)$, so is $(\mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \mathbf{T}^{-1}\mathbf{B}, \mathbf{C}\mathbf{T}, \mathbf{D})$. One particular realization that is of immediate interest to us is the balanced realization, as introduced next.

Definition 2.5 For a rational stable system $\mathbf{G}(s)$, the state-space realization $\mathbf{G}(s) \leftrightarrow (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is called a balanced realization, if $\mathbf{X}_H, \mathbf{Y}_H \succeq 0$ that solve the Lyapunov equations (2.13)-(2.14) are diagonal and equal [42, 117].

As it turns out that for the balanced realization, the controllability and observability gramians are equal to $\text{diag}(\sigma_{Hi}(\mathbf{G}(s)))$, *i.e.*, the matrix containing the Hankel singular values as its diagonal elements. Any rational stable system admits a balanced realization and an algorithm for the construction of balanced realization is available in [117]. The balanced realization is frequently used in obtaining approximate low order models for a system with a large number of states [42].

For later development in this chapter, we derive the balanced state-space realization of the Blaschke product $\mathcal{B}_{po}^{-1}(s)$. For notational simplicity, we consider that the number of unstable poles, $n_p \leq 2$, which can be easily extended to systems with $n_p > 2$ by induction.

A similar method has been used by Chen [22] earlier for finding the balanced realization of $\mathcal{B}_{zi}(s)$.

Let $\mathcal{B}_{po}^{-1}(s) = \mathcal{B}_{p_2}^{-1}\mathcal{B}_{p_1}^{-1}(s)$. Using (2.4), the balanced realization of $\mathcal{B}_{p_i}^{-1}(s)$ is given as $\mathcal{B}_{p_i}^{-1}(s) \leftrightarrow (\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i, \mathbf{D}_i)$, where

$$\mathbf{A}_i = p_i \quad \mathbf{B}_i = -\sqrt{2\mathrm{Re}(p_i)}\,\hat{\mathbf{y}}_{p_i}^* \quad \mathbf{C}_i = \sqrt{2\mathrm{Re}(p_i)}\,\hat{\mathbf{y}}_{p_i} \quad \mathbf{D}_i = \mathbf{I} \tag{2.15}$$

Using (2.15), the balanced realization of $\mathcal{B}_{po}^{-1}(s)$ is given as $\mathcal{B}_{po}^{-1}(s) \leftrightarrow (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} \mathbf{A}_2 & \mathbf{B}_2\mathbf{C}_1 \\ \mathbf{0} & \mathbf{A}_1 \end{bmatrix} = \begin{bmatrix} p_2 & 2\sqrt{\mathrm{Re}(p_1)\mathrm{Re}(p_2)}\,\hat{\mathbf{y}}_{p_2}^*\hat{\mathbf{y}}_{p_1} \\ \mathbf{0} & p_1 \end{bmatrix} \\
\mathbf{B} &= \begin{bmatrix} \mathbf{B}_2\mathbf{D}_1 \\ \mathbf{B}_1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2\mathrm{Re}(p_2)}\,\hat{\mathbf{y}}_{p_2}^* \\ -\sqrt{2\mathrm{Re}(p_1)}\,\hat{\mathbf{y}}_{p_1}^* \end{bmatrix} \\
\mathbf{C} &= \begin{bmatrix} \mathbf{C}_2 & \mathbf{D}_2\mathbf{C}_1 \end{bmatrix} = \begin{bmatrix} \sqrt{2\mathrm{Re}(p_2)}\,\hat{\mathbf{y}}_{p_2} & \sqrt{2\mathrm{Re}(p_1)}\,\hat{\mathbf{y}}_{p_1} \end{bmatrix} \\
\mathbf{D} &= \mathbf{D}_2\mathbf{D}_1 = \mathbf{I}
\end{aligned} \tag{2.16}$$

2.3 Problem Formulation and Simplification

In this section, we formulate an optimal controller design problem that minimizes input usage for stabilization. It is shown how the general results on optimal control can be simplified when only input performance is considered. This simplification in turn enable us to explicitly characterize the achievable input performance.

Consider the system shown in Figure 2.3, where all exogenous inputs, *e.g.* load change, measurement noise, set point change, have been collected in the block $\mathbf{G}_w(s)$. The closed loop transfer matrix from disturbances to inputs is given as,

$$\mathbf{T}_{uw}(s) = \mathbf{W}_u\mathbf{K}(s)\,(\mathbf{I} + \mathbf{G}\mathbf{K}(s))^{-1}\,\mathbf{G}_w(s) \tag{2.17}$$

The objective is to characterize the minimal input usage required for stabilization expressed in terms of the norm of $\mathbf{T}_{uw}(s)$ as:

$$I_i = \|\mathbf{W}_u\mathbf{K}(s)\,(\mathbf{I} + \mathbf{G}\mathbf{K}(s))^{-1}\,\mathbf{G}_w(s)\|_i \quad i = 2, \infty \tag{2.18}$$

Assumption 2.2 We make the following assumptions:

(a) $\mathbf{G}(s)$ is strictly proper.

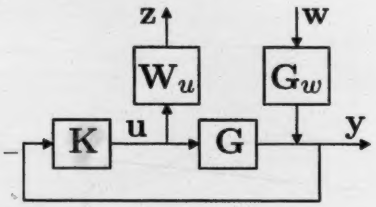

Figure 2.3: Closed loop system for characterization of achievable input performance

(b) $\mathbf{W}_u(s)$ is left invertible and (if unstable) has the same unstable poles as $\mathbf{G}(s)$ with the associated input pole directions.

(c) $\mathbf{G}_w(s)$ is right-invertible and (if unstable) has the same unstable poles as $\mathbf{G}(s)$ with the associated output pole directions.

Assumption 2.2(a) is made for notational simplicity and the extension to the general case is simple (see [117, p.261] for details). The left and right invertibility of $\mathbf{W}_u(s)$ and $\mathbf{G}_w(s)$ respectively ensures that the optimal controller design problem is nonsingular.

To illustrate the necessity of $\mathbf{W}_u(s)$ and $\mathbf{G}_w(s)$ having the same unstable poles as $\mathbf{G}(s)$ with the associated input and output pole directions respectively, consider that $\mathbf{W}_u(s) = \mathbf{I}$ and $\mathbf{G}_w(s)$ has a single unstable pole p_w such that $\mathbf{G}_w^{-1}(p_w)\mathbf{y}_{p_w} = \mathbf{0}$. Let $\{p_i\} \in \mathbb{C}^{n_p}$ be the unstable poles of $\mathbf{G}(s)$ such that $\mathbf{G}^{-1}(p_i)\mathbf{y}_{p_i} = \mathbf{0}$. For internal stability, the unstable poles of $\mathbf{G}(s)$ and $\mathbf{GK}(s)$ are the same and

$$\begin{aligned}
\mathbf{K}^{-1}\mathbf{G}^{-1}(p_i)\mathbf{y}_{p_i} &= \mathbf{0} \\
\left(\mathbf{I} + \mathbf{K}^{-1}\mathbf{G}^{-1}(p_i)\right)\mathbf{y}_{p_i} &= \mathbf{y}_{p_i} \\
\mathbf{GK}(p_i)\left(\mathbf{I} + \mathbf{GK}(p_i)\right)^{-1}\mathbf{y}_{p_i} &= \mathbf{y}_{p_i} \\
\mathbf{K}(p_i)\left(\mathbf{I} + \mathbf{GK}(p_i)\right)^{-1}\mathbf{y}_{p_i} &= \mathbf{G}^{-1}(p_i)\mathbf{y}_{p_i} = \mathbf{0} \qquad (2.19)
\end{aligned}$$

It follows from (2.19) that the locations of RHP zeros and output zero directions of $\mathbf{K}(s)\left(\mathbf{I} + \mathbf{GK}(s)\right)^{-1}$ are the same as the locations of the RHP poles and input pole directions of $\mathbf{G}(s)$. Defining the sensitivity function as $\mathbf{S}(s) = \left(\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)\right)^{-1}$ and using results on Blaschke products (2.2) and (2.5),

$$\begin{aligned}
\mathbf{KSG}_w(s) &= \left[\mathbf{KS}(s)\right]_{mi}\mathcal{B}_{zi}[\mathbf{KS}(s)]\mathcal{B}_{po}^{-1}[\mathbf{G}_w(s)]\left[\mathbf{G}_w(s)\right]_{so} \\
&= \left[\mathbf{KS}(s)\right]_{mi}\mathcal{B}_{po}[\mathbf{G}(s)]\mathcal{B}_{po}^{-1}[\mathbf{G}_w(s)]\left[\mathbf{G}_w(s)\right]_{so}
\end{aligned}$$

If the controller is designed to stabilize $\mathbf{KS}(s)$, the stability of $\mathbf{T}_{uw}(s)$ depends on the stability of $\mathcal{B}_{po}[\mathbf{G}(s)]\mathcal{B}_{po}^{-1}[\mathbf{G}_w(s)]$. Since the Blaschke products can be calculated for any permutation of poles and zeros, $\mathcal{B}_{po}[\mathbf{G}(s)]\mathcal{B}_{po}^{-1}[\mathbf{G}_w(s)]$ is stable iff $p_w = p_i$ and $\mathbf{y}_{p_w} = \mathbf{y}_{p_i}$ for some i. Similar conclusions can be drawn when $\mathbf{G}_w(s)$ has more than one unstable pole or when $\mathbf{W}_u(s)$ is also unstable.

With Assumption 2.2, Let $\mathbf{W}_u(s)$ and $\mathbf{G}_w(s)$ be factorized as

$$\begin{aligned}\mathbf{W}_u(s) &= \mathcal{B}_{po}^{-1}[\mathbf{W}_u(s)]\,\mathcal{B}_{zo}[\mathbf{W}_u(s)]\,[\mathbf{W}_u(s)]_{sm}\\ \mathbf{G}_w(s) &= [\mathbf{G}_w(s)]_{sm}\,\mathcal{B}_{pi}^{-1}[\mathbf{G}_w(s)]\,\mathcal{B}_{zi}[\mathbf{G}_w(s)]\end{aligned}$$

where $[\mathbf{W}_u(s)]_{sm}$ and $[\mathbf{G}_w(s)]_{sm}$ are the stable minimum-phase parts of $\mathbf{W}_u(s)$ and $\mathbf{G}_w(s)$ respectively. Define

$$\begin{aligned}\hat{\mathbf{G}}(s) &= [\mathbf{G}_w(s)]_{sm}^{-1}\,\mathbf{G}(s)\,[\mathbf{W}_u(s)]_{sm}^{-1}\\ \hat{\mathbf{K}}(s) &= [\mathbf{W}_u(s)]_{sm}\,\mathbf{K}(s)\,[\mathbf{G}_w(s)]_{sm}\end{aligned} \tag{2.20}$$

where $\hat{\mathbf{G}}(s)$ is an $n_y \times n_u$ dimensional transfer matrix. It follows from (2.17) that

$$\begin{aligned}I_i = \|\mathcal{B}_{po}^{-1}[\mathbf{W}_u(s)]\,\mathcal{B}_{zo}[\mathbf{W}_u(s)]\,\hat{\mathbf{K}}(s)(\mathbf{I}+\hat{\mathbf{G}}\hat{\mathbf{K}}(s))^{-1}\cdot&\\ \mathcal{B}_{pi}^{-1}[\mathbf{G}_w(s)]\,\mathcal{B}_{zi}[\mathbf{G}_w(s)]\,\|_i \quad i = 2,\infty&\end{aligned} \tag{2.21}$$

By simplifying (2.21),

$$I_i = \|\hat{\mathbf{K}}(s)(\mathbf{I}+\hat{\mathbf{G}}(s)\hat{\mathbf{K}}(s))^{-1}\|_i \quad i = 2,\infty \tag{2.22}$$

We point out that in (2.22), $\mathcal{B}_{po}^{-1}[\mathbf{W}_u(s)]$ and $\mathcal{B}_{pi}^{-1}[\mathbf{G}_w(s)]$ can be factored out without jeopardizing the internal stability, only when Assumptions 2.2(b)-(c) are satisfied. Now, $\|\mathbf{T}_{uw}(s)\|_i$, $i = 2,\infty$ is minimized by designing an optimal controller for $\hat{\mathbf{G}}(s)$, where the following are equivalent: (a) $\hat{\mathbf{K}}(s)$ stabilizes $\hat{\mathbf{G}}(s)$, and (b) $\mathbf{K}(s)$ stabilizes $\mathbf{G}(s)$. In the remaining discussion, we treat $\hat{\mathbf{G}}(s)$ as the system without loss of generality. These manipulations further allows us to represent the generalized plant as

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}}+\hat{\mathbf{B}}\mathbf{u}\\ \mathbf{y} &= \hat{\mathbf{C}}\hat{\mathbf{x}}+\mathbf{w}\\ \mathbf{z} &= \mathbf{u}\end{aligned} \tag{2.23}$$

where $\hat{\mathbf{G}}(s) \leftrightarrow (\hat{\mathbf{A}},\hat{\mathbf{B}},\hat{\mathbf{C}})$. Notice that we have transformed a controller design problem where the closed loop system is driven by disturbances filtered through an arbitrary disturbance model to an equivalent problem, in which the closed loop system is driven by measurement noise only. The latter problem is much simpler to solve, as demonstrated later in this section.

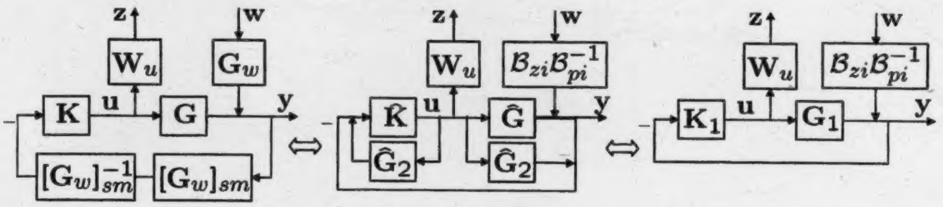

Figure 2.4: Simplifying transformations on the closed loop system

For the system (2.23), let $\hat{\mathbf{X}}_2$, $\hat{\mathbf{Y}}_2$ and $\hat{\mathbf{X}}_\infty$, $\hat{\mathbf{Y}}_\infty$ be the solutions of corresponding AREs for the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ optimal control (see § 2.2.3). By comparing (2.23) with (2.6), we notice that for the system (2.23), the corresponding AREs for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal controller design are the same. It follows that $\hat{\mathbf{X}}_2 = \hat{\mathbf{X}}_\infty = \hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}_2 = \hat{\mathbf{Y}}_\infty = \hat{\mathbf{Y}}$. This observation in turn implies that $\hat{\mathbf{F}}_2 = \hat{\mathbf{F}}_\infty = \hat{\mathbf{F}}$ and $\hat{\mathbf{L}}_2 = \hat{\mathbf{L}}_\infty = \hat{\mathbf{L}}$.

Let $\mathbf{T}$ be a state transformation matrix such that $\mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T} = \mathrm{diag}(\mathbf{P}_s, \mathbf{P})$, where $\mathbf{P}_s$ and $\mathbf{P}$ contain all the stable and unstable modes respectively. Rearranging and partitioning the states of the transformed system

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{T}^{-1}\hat{\mathbf{B}}\mathbf{u} = \begin{bmatrix} \mathbf{P}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{bmatrix}\tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{B}_s \\ \mathbf{B} \end{bmatrix}\mathbf{u} \\ \mathbf{y} &= \hat{\mathbf{C}}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{w} = \begin{bmatrix} \mathbf{C}_s & \mathbf{C} \end{bmatrix}\tilde{\mathbf{x}} + \mathbf{w} \qquad (2.24)\end{aligned}$$

Let $\tilde{\mathbf{X}} = \mathbf{T}^{-1}\hat{\mathbf{X}}\mathbf{T}$ and $\tilde{\mathbf{Y}} = \mathbf{T}^{-1}\hat{\mathbf{Y}}\mathbf{T}$ solve the corresponding AREs for the transformed system (2.24). Then, to be non-negative definite, $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ must assume the form

$$\tilde{\mathbf{X}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{bmatrix} \quad \tilde{\mathbf{Y}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix}$$

where $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^{n_p \times n_p} \succ 0$ and it suffices to solve

$$\mathbf{XP} + \mathbf{P}^*\mathbf{X} - \mathbf{XBB}^*\mathbf{X} = 0 \qquad (2.25)$$

$$\mathbf{YP}^* + \mathbf{PY} - \mathbf{YC}^*\mathbf{CY} = 0 \qquad (2.26)$$

Let $\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}_1(s) + \hat{\mathbf{G}}_2(s)$ such that $\hat{\mathbf{G}}_1(s) = \mathcal{U}(\hat{\mathbf{G}}(s))$ and $\hat{\mathbf{G}}_2(s) \in \mathcal{RH}_\infty$, where $\mathcal{U}(\hat{\mathbf{G}}(s))$ is the unstable part of $\hat{\mathbf{G}}(s)$. The triplet $(\mathbf{P}, \mathbf{B}, \mathbf{C})$ can be seen as the realization of $\hat{\mathbf{G}}_1(s)$ and (2.25)-(2.26) as the corresponding AREs for $\hat{\mathbf{G}}_1(s)$. Then the achievable input performance depends only on the unstable part of the system. This is further illustrated by defining $\hat{\mathbf{K}}(s) = \hat{\mathbf{K}}_1(s)(\mathbf{I} - \hat{\mathbf{G}}_2\hat{\mathbf{K}}_1(s))^{-1}$. With this parametrization of $\hat{\mathbf{K}}(s)$,

$$\hat{\mathbf{K}}(s)(\mathbf{I} - \hat{\mathbf{G}}\hat{\mathbf{K}}(s))^{-1} = \hat{\mathbf{K}}_1(s)(\mathbf{I} - \hat{\mathbf{G}}_2\hat{\mathbf{K}}_1(s))^{-1}$$

Thus $\hat{\mathbf{K}}(s)$ exactly cancels the stable part of the system. The different transformations used in this section and their equivalence are shown in Figure 2.4.

For the transformed system (2.24), the state feedback and the output injection matrices are given as,

$$\begin{aligned} \tilde{\mathbf{F}} &= \hat{\mathbf{F}}\mathbf{T} = [\,\mathbf{0} \;\; \mathbf{F}\,] = [\,\mathbf{0} \;\; -\mathbf{B}^*\mathbf{X}\,] && (2.27) \\ \tilde{\mathbf{L}} &= \mathbf{T}^*\hat{\mathbf{L}} = [\,\mathbf{0} \;\; \mathbf{L}\,]' = [\,\mathbf{0} \;\; -\mathbf{Y}\mathbf{C}^*\,]' && (2.28) \end{aligned}$$

By substituting for $\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}, \tilde{\mathbf{F}}$ and $\tilde{\mathbf{L}}$ in (2.8) and (2.12), the expressions for achievable input performance can be simplified as,

$$\begin{aligned} I_2^2 &= \mathrm{tr}(\mathbf{F}\mathbf{Y}\mathbf{F}^*) = \mathrm{tr}(\mathbf{L}^*\mathbf{X}\mathbf{L}) && (2.29) \\ I_\infty &= \rho^{\frac{1}{2}}(\mathbf{X}\mathbf{Y}) && (2.30) \end{aligned}$$

The equations (2.25) and (2.26) form the cornerstone for much of the remaining development in this chapter. In general, for $\mathcal{H}_\infty$ optimal control, the resulting AREs are dependent on γ and thus need to be solved iteratively. In contrast, the expressions (2.25)-(2.26) are independent of γ and can be solved directly. Further note that when (2.25) and (2.26) are pre- and post-multiplied by $\mathbf{X}^{-1}$ and $\mathbf{Y}^{-1}$, the resulting expressions are similar to Lyapunov equations. When all the unstable poles of the system are distinct, a closed form solution of (2.25)-(2.26) can be derived, which is expressed in terms of the unstable poles and the matrices $\mathbf{B}$ and $\mathbf{C}$ only.

For a system with distinct unstable poles, we can select the state transformation matrix $\mathbf{T}$ such that $\mathbf{P}$ is diagonal and is given as $\mathbf{P} = \mathrm{diag}(p_1, \cdots, p_{n_p})$, $\mathrm{Re}(p_i) > 0$. Let the Hermitian matrix $\mathbf{M} \in \mathbb{C}^{n_p \times n_p}$ be defined as

$$[m_{ij}] = 1/(p_i + p_j^*) \tag{2.31}$$

Lemma 2.1 For a system with distinct poles, let $\mathbf{X}, \mathbf{Y} \succ 0$ solve the AREs (2.25)-(2.26) and $\mathbf{M}$ be given by (2.31). Then

$$\begin{aligned} \mathbf{X}^{-1} &= \sum_{i=1}^{n_u} \mathrm{diag}(\mathbf{B}_i)\; \mathbf{M}\; \mathrm{diag}(\mathbf{B}_i)^* && (2.32) \\ \mathbf{Y}^{-1} &= \sum_{j=1}^{n_y} \mathrm{diag}(\mathbf{C}_j')^*\; \mathbf{M}\; \mathrm{diag}(\mathbf{C}_j') && (2.33) \end{aligned}$$

Proof: Pre- and post-multiplying (2.25) by $\mathbf{X}^{-1}$ gives

$$\mathbf{P}\mathbf{X}^{-1} + \mathbf{X}^{-1}\mathbf{P}^* = \mathbf{B}\mathbf{B}^* \tag{2.34}$$

Then [63], $\mathbf{X}^{-1} = \mathbf{M} \circ (\mathbf{B}\mathbf{B}^*)$, where $\circ$ is the Hadamard or element-wise product. Noting that $\mathbf{B}\mathbf{B}^* = \sum_{i=1}^{n_u} \mathbf{B}_i\mathbf{B}_i^*$,

$$\mathbf{X}^{-1} = \sum_{i=1}^{n_u} \mathbf{M} \circ (\mathbf{B}_i\mathbf{B}_i^*)$$

and (2.32) follows. Equation (2.33) follows from a dual argument. ■

2.4 SISO systems

In this section, we quantify achievable input performance of SISO systems with and without time delay. It is assumed that all the unstable poles of the system are distinct. With this assumption, the expressions for the achievable input performance can be expressed in terms of the unstable poles and the matrices $\mathbf{B}$ and $\mathbf{C}$ only. The general case is considered in the next section.

2.4.1 Rational Systems

We derive the expressions for achievable input performance for rational SISO systems next. The usefulness of these expressions is demonstrated using a process design example. These results also form the basis for derivation of similar expressions for SISO systems with time delay.

Lemma 2.2 For M defined by (2.31), let $p_i \neq p_j$ for all $i, j = 1 \cdots n_p$. Then $\mathbf{M}^{-1}$ is given as

$$[\mathbf{M}^{-1}]_{ij} = \frac{(p_i^* + p_i)(p_j + p_j^*)}{p_i^* + p_j} \left(\prod_{\substack{k=1 \\ k \neq i}}^{n_p} \frac{(p_i^* + p_k)}{(p_i^* - p_k^*)} \right) \left(\prod_{\substack{k=1 \\ k \neq j}}^{n_p} \frac{(p_j + p_k^*)}{(p_j - p_k)} \right)$$

Lemma 2.2 is easily verified by evaluating $\mathbf{M}\mathbf{M}^{-1}$ or $\mathbf{M}^{-1}\mathbf{M}$. Note for SISO systems, $\mathbf{b} = [b_i]$, $\mathbf{b} = [c_j]$.

Proposition 2.1 For a rational SISO system $g(s)$ with distinct poles, let $\mathcal{U}(g(s)) \leftrightarrow (\mathbf{P}, \mathbf{b}, \mathbf{c})$ such that $\mathbf{P} = \mathrm{diag}(p_1 \cdots p_{n_p})$, $\mathrm{Re}(p_i) > 0$. Then

$$I_2^2 = \left[\frac{|\mathbf{q}_i|^2}{b_i c_i}\right] \mathbf{M} \left[\frac{|\mathbf{q}_i|^2}{b_i^* c_i^*}\right]^T \qquad (2.35)$$

$$I_\infty^2 = |\underline{\lambda}^{-1}(\mathrm{diag}(b_i^* c_i^*)\ \mathbf{M}\ \mathrm{diag}(b_i c_i)\ \mathbf{M})| \qquad (2.36)$$

where $\mathbf{M}$ is defined by (2.31) and $\mathbf{q}_i$ is the sum of i^{th} column of $\mathbf{M}^{-1}$ or $\mathbf{q} = \mathbf{1}_{n_p}^T \mathbf{M}^{-1}$.

Proof: (1) For (2.35), substituting for $\mathbf{X}$ and $\mathbf{Y}$ in (2.29) using Lemma 2.1,

$$\begin{aligned} I_2^2 &= \mathbf{f}\mathbf{Y}\mathbf{f}^* = \mathbf{b}^*\mathbf{X}\mathbf{Y}\mathbf{X}\mathbf{b} \\ &= \mathbf{1}_{n_p}^T\mathbf{M}^{-1}\,(\mathrm{diag}(\mathbf{b})\mathrm{diag}(\mathbf{c}))^{-1}\,\mathbf{M}^{-1}(\mathrm{diag}(\mathbf{b}^*)\mathrm{diag}(\mathbf{c}^*))^{-1}\,\mathbf{M}^{-1}\,\mathbf{1}_{n_p} \qquad (2.37) \end{aligned}$$

Based on Lemma 2.2,

$$\mathbf{q}_i = (p_i + p_i^*)\prod_{\substack{k=1\\k\neq i}}^{n_p}\frac{(p_i + p_k^*)}{(p_i - p_k)};\quad i = 1\cdots n_p \qquad (2.38)$$

and $\mathbf{M}^{-1} = \mathrm{diag}(\mathbf{q}^*)\mathbf{M}\mathrm{diag}(\mathbf{q})$. By substituting for $\mathbf{M}^{-1}$ and $\mathbf{1}_{n_p}^T\mathbf{M}^{-1}$, (2.37) can be simplified as,

$$I_2^2 = \mathbf{q}\,(\mathrm{diag}(\mathbf{b})\mathrm{diag}(\mathbf{c}))^{-1}\,\mathrm{diag}(\mathbf{q}^*)\,\mathbf{M}\,\mathrm{diag}(\mathbf{q})\,(\mathrm{diag}(\mathbf{b}^*)\mathrm{diag}(\mathbf{c}^*))^{-1}\,\mathbf{q}^*$$

The equation (2.35) can be now obtained by simplifying the above expression using the identity $\mathbf{q}_i\mathbf{q}_i^* = |\mathbf{q}_i|^2$.

(2) For (2.36),

$$I_\infty^2 = \rho(\mathbf{X}\mathbf{Y}) = \left|\underline{\lambda}^{-1}(\mathbf{Y}^{-1}\mathbf{X}^{-1})\right|$$

By substituting for $\mathbf{X}^{-1}$ and $\mathbf{Y}^{-1}$ using Lemma 2.1

$$\begin{aligned} I_\infty^2 &= \left|\underline{\lambda}^{-1}(\mathrm{diag}(\mathbf{c}^*)\,\mathbf{M}\,\mathrm{diag}(\mathbf{c})\,\mathrm{diag}(\mathbf{b})\,\mathbf{M}\,\mathrm{diag}(\mathbf{b})^*)\right| \\ &= \left|\underline{\lambda}^{-1}(\mathrm{diag}(\mathbf{b})^*\,\mathrm{diag}(\mathbf{c}^*)\,\mathbf{M}\,\mathrm{diag}(\mathbf{c})\,\mathrm{diag}(\mathbf{b})\,\mathbf{M})\right| \\ &= \left|\underline{\lambda}^{-1}(\mathrm{diag}(b_i^*c_i^*)\,\mathbf{M}\,\mathrm{diag}(b_ic_i)\,\mathbf{M})\right| \end{aligned}$$

■

In the realization, $\mathcal{U}(g(s)) \leftrightarrow (\mathbf{P}, \mathbf{b}, \mathbf{c})$, when $g(s)$ has only real unstable poles only, $\mathbf{b}^* = \mathbf{b}$ and $\mathbf{c}^* = \mathbf{c}$. In this case, (2.36), can be further simplified as,

$$\begin{aligned} I_\infty^2 &= \underline{\lambda}^{-1}\left((\mathrm{diag}(b_ic_i)\mathbf{M})^2\right) \\ I_\infty &= \left|\underline{\lambda}^{-1}(\mathrm{diag}(b_ic_i)\mathbf{M})\right| \end{aligned}$$

Remark 2.1 The expression for $\mathbf{q}$ in (2.38) appears to suggest that in general, $I_2 \to \infty$ as $p_i \to p_j$ for some i, j, which is clearly not true. Since $b_ic_i = [\hat{g}(s)(s - p_i)]_{s=p_i}$, $b_ic_i \to \infty$, as $p_i \to p_j$, which negates the effect of $\mathbf{q}$. But when the system has an RHP zero close to RHP poles, b_ic_i fails to increase monotonically and stabilization can be difficult. For example, consider $\hat{g}(s) = \frac{(s-p)}{(s-p+\epsilon)(s-p-\epsilon)}$. As $\epsilon \to 0$, the RHP poles approach the zero. Due to near cancellation of the unstable pole by the zero, $I_2, I_\infty \to \infty$ as $\epsilon \to 0$.

Example 2.1 In order to demonstrate the utility of Proposition 2.1 for process design purposes, consider a rational SISO system with two distinct unstable poles $p_1, p_2 \in \mathbb{R}$ and a RHP zero z. The location of z can be influenced by process or operating point changes. The objective is to choose z in the range $0 < z \leq \max(p_1, p_2)$ such that input usage for stabilization is minimal. A pure numerical approach requires solving the following nested optimization problem:

$$\min_{z} \inf_{k(s)} \|k(s)(1 + gk(s))^{-1}\|_i \quad i = 2, \infty$$

Using Proposition 2.1, the optimal value of z can be characterized explicitly. As $z \longrightarrow p_i$, the joint controllability and observability of p_i reduces monotonically increasing the input requirement. Notice that

$$b_1 c_1 = \frac{z - p_1}{p_1 - p_2} \qquad b_2 c_2 = \frac{z - p_2}{p_2 - p_1}$$

Using (2.35) and (2.39),

$$\begin{aligned} I_2^2 &= \frac{8(p_1 + p_2)^3 \left[p_1^2(p_2 - z)^2 + p_2^2(p_1 - z)^2 + p_1 p_2(3z^2 - p_1 p_2)\right]}{(p_1 - z)^2 + (p_2 - z)^2} \\ I_\infty &= \frac{4p_1 p_2(p_1 + p_2)}{z(p_1 + p_2) - \left[p_1^2(2p_2 - z)^2 + p_2^2(2p_1 - z)^2 + 2p_1 p_2(3z^2 - 2p_1 p_2)\right]^{0.5}} \end{aligned}$$

The optimal value of z in the range $0 < z \leq \max(p_1, p_2)$ can be obtained by evaluating the stationary points of (2.35) and (2.36),

$$\begin{aligned} z_{\mathcal{H}_2,\text{opt}} &= \frac{p_1 p_2 \left(3(p_1 + p_2) \pm \sqrt{5p_1^2 + 5p_2^2 + 6p_1 p_2}\right)}{2(p_1^2 + p_2^2 + 3p_1 p_2)} \\ z_{\mathcal{H}_\infty,\text{sub}} &= \frac{4p_1 p_2(p_1 + p_2)}{p_1^2 + p_2^2 + 6p_1 p_2} \end{aligned}$$

2.4.2 Time Delay Systems

Many systems arising in practice contain time delay. These irrational systems cannot be handled directly in the optimal control framework discussed in § 2.2.3. A common approach for optimal control for such systems is to design the controller based on a rational approximation (*e.g.* Páde approximation) of the time delay system. In this thesis, we use this approach and the achievable performance is characterized by letting the order of approximation approach infinity in the limit.

To extend Proposition 2.1 to systems with a finite time delay, let $\hat{g}(s)$ be expressed as,

$$\hat{g}(s) = \tilde{g}(s)e^{-\theta s} \tag{2.39}$$

where $\tilde{g}$ is the delay-free part of the system. If $g_w(s)$ also contains delay, the delay can be factored as an all-pass factor and thus $\hat{g}(s)$ remains causal (*cf.* (2.20)).

Lemma 2.3 Consider $\mathbf{H}(s) \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$ such that $\mathbf{P} = \text{diag}(p_1 \cdots p_{n_p})$, $\text{Re}(p_i) > 0$, $p_i \neq p_j$. Let $\mathbf{H_1}(\mathrm{s}) \in \mathcal{RH}_\infty$ with no zeros at p_i. Then

$$\mathcal{U}(\mathbf{H}_1(s)\mathbf{H}(s)) = \sum_{i=1}^{n_p} \frac{1}{s - p_i} \mathbf{H}_1(p_i)\mathbf{C}_i\mathbf{B}_i^{'} \tag{2.40}$$

Proof: Using dyadic expansion of $\mathbf{H}(s)$,

$$\mathbf{H}(s) = \sum_{i=1}^{n_p} \frac{1}{s - p_i} \mathbf{C}_i\mathbf{B}_i^{'}$$

Let $\mathcal{U}(\mathbf{H}_1(s)\mathbf{H}(s)) \leftrightarrow (\tilde{\mathbf{P}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$. Since $\mathbf{H_1}(\mathrm{s})$ does not cancel RHP poles of $\mathbf{H}(s)$, $\tilde{\mathbf{P}} = \mathbf{P}$. Now, $\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i^{'} = [\mathbf{H}_1(s)\mathbf{H}(s)(s - p_i)]_{s=p_i}$ and (2.40) follows. ■

Note that the applicability of Lemma 2.3 is not limited to the case where all modes of $\mathbf{H}(s)$ are unstable, since $\mathcal{U}(\mathbf{H}_1(s)\mathbf{H}(s)) = \mathcal{U}(\mathbf{H}_1(s)\mathcal{U}(\mathbf{H}(s)))$.

Proposition 2.2 Let the SISO system expressed by (2.39) have distinct unstable poles and $\mathcal{U}(\tilde{g}(s)) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}})$ such that $\mathbf{P} = \text{diag}(p_1 \cdots p_{n_p})$, $\text{Re}(p_i) > 0$ and $\mathbf{\Gamma} = \text{diag}(e^{\theta p_1} \cdots e^{\theta p_{n_p}})$. Then

$$\begin{aligned} I_2^2 &= \left[\frac{|\mathbf{q}_i|^2}{\tilde{b}_i\tilde{c}_i}\right] \mathbf{\Gamma}\mathbf{M}\mathbf{\Gamma}^* \left[\frac{|\mathbf{q}_i|^2}{\tilde{b}_i^*\tilde{c}_i^*}\right]^{'} \quad i = 1 \cdots n_p & (2.41) \\ I_\infty^2 &= |\underline{\lambda}^{-1}(\mathbf{\Gamma}^{-*}\text{diag}(\tilde{b}_i^*\tilde{c}_i^*)\mathbf{M}\mathbf{\Gamma}^{-1}\text{diag}(\tilde{b}_i\tilde{c}_i)\mathbf{M})| & (2.42) \end{aligned}$$

where $\mathbf{M}$ is defined by (2.31) and $\mathbf{q} = \mathbf{1}_{n_p}^{'}\mathbf{M}^{-1}$.

Proof: Let $f(\theta s, n)$ be the n^{th} order rational approximation of $e^{-\theta s}$ (*e.g.* Páde approximation). For any n, if a RHP zero of $f(\theta s, n)$ cancels a RHP pole of $\tilde{\mathbf{G}}(s)$, the system is not stabilizable due to presence of hidden unstable modes. However, as $n \to \infty$, the magnitude of RHP zeros of $f(\theta s, n)$ approaches infinity. Thus, for an FDLTI system with poles at finite locations, such cancellation of RHP pole of $\tilde{\mathbf{G}}(s)$ by an RHP zero of $f(\theta s, n)$ does not occur for all $n \geq N$ for sufficiently large N.

(1) For (2.41), using (2.40), $b_i c_i \approx \tilde{b}_i\tilde{c}_i f(\theta p_i, n)$, $n \geq N$ and

$$\begin{aligned} I_2^2(n) &= \left[\frac{|\mathbf{q}_i|^2}{\tilde{b}_i\tilde{c}_i f(\theta p_i, n)}\right] \mathbf{M} \left[\frac{|\mathbf{q}_i|^2}{\tilde{b}_i^*\tilde{c}_i^* f(\theta p_i, n)}\right]^{'} \\ &= \sum_{i=1}^{n_p}\sum_{j=1}^{n_p} \frac{|\mathbf{q}_i|^2}{\tilde{b}_i\tilde{c}_i}\frac{|\mathbf{q}_j|^2}{\tilde{b}_j\tilde{c}_j} m_{ij} f^{-1}(\theta p_i, n) f^{-1}(\theta p_j, n) & (2.43) \end{aligned}$$

As $\lim_{n\to\infty} f(\theta p_i, s) = e^{-\theta s}$. Then, $\lim_{n\to\infty} f^{-1}(\theta p_i, n) = e^{\theta p_i}$ and $\lim_{n\to\infty} f^{-1}(\theta p_i, n) f^{-1}(\theta p_j, n) = e^{\theta p_i} e^{\theta p_j}$. Noting that except the bilinear term $f^{-1}(\theta p_i, n) f^{-1}(\theta p_j, n)$, all other terms in (2.43) are independent of n, we conclude that $\lim_{n\to\infty} I_2^2(n)$ exists and is given by (2.41).

(2) For (2.42), using similar arguments as before and following the proof of Proposition 2.1,

$$I_\infty^2(n) = \left|\underline{\lambda}^{-1}(\mathrm{diag}(f(\theta p_i, n))^{-*}\ \mathrm{diag}(\tilde{b}_i^* \tilde{c}_i^*)\ \mathbf{M}\ \mathrm{diag}(f(\theta p_i, n))^{-1}\ \mathrm{diag}(\tilde{b}_i \tilde{c}_i)\mathbf{M})\right|$$

The eigenvalues are roots of a polynomial equation, whose coefficients are functions of $f^{-1}(\theta p_i, n)$. As $n \to \infty$, these coefficients and thus the roots converge. Hence, $\lim_{n\to\infty} I_\infty^2(n)$ exists and is given by (2.42). ■

Similar to (2.39), for a system with real unstable poles only, (2.42) can be simplified to

$$I_\infty^2 = \left|\underline{\lambda}^{-1}(\Gamma^{-1}\mathrm{diag}(b_i c_i)\mathbf{M})\right|$$

By differentiating (2.41) with respect to θ,

$$\begin{aligned} \frac{dI_2^2}{d\theta} &= \sum_{i=1}^{n_p}\sum_{j=1}^{n_p} p_i p_j \frac{|\mathbf{q}_i|^2}{\tilde{b}_i \tilde{c}_i} \frac{|\mathbf{q}_j|^2}{\tilde{b}_j \tilde{c}_j} m_{ij} e^{p_i\theta} e^{p_j\theta} \\ &\geq \min_i p_i^2 I_2^2 \end{aligned}$$

Thus, $dI_2/d\theta > 0$ for all θ. Similar conclusions can be drawn by differentiating I_∞ with respect to θ. This shows that for SISO systems, the input usage cannot be decreased by introducing additional lag in the system. Surprisingly, for MIMO systems, such an intuitive conclusion does not hold, as is shown later.

Corollary 2.1 Under the same conditions as Proposition 2.2, let $g_p(s) \leftrightarrow (\mathbf{P}, \mathbf{\Gamma}^{-1}\tilde{\mathbf{b}}, \tilde{\mathbf{c}})$ or $(\mathbf{P}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}\mathbf{\Gamma}^{-1})$. Then $I_2(\hat{g}(s)) = I_2(g_p(s))$ and $I_\infty(\hat{g}(s)) = I_\infty(g_p(s))$.

It follows from corollary 2.1 that I_2 and I_∞ for a time delay system depend on its unstable projection, which is rational.

Corollary 2.2 For a SISO system with a single real unstable pole p,

$$I_2^2 = \frac{8p^3 e^{2p\theta}}{\tilde{b}^2 \tilde{c}^2} \qquad I_\infty = \frac{2pe^{p\theta}}{|\tilde{b}\tilde{c}|} \tag{2.44}$$

Corollary 2.2 can be shown to be true by considering (2.41) and noting that in this case $\tilde{\mathbf{b}}$, $\tilde{\mathbf{c}}$ are scalars and $\mathbf{M} = 1/2p$. For delay-free systems, Havre and Skogestad [58] earlier obtained expressions similar to (2.44). Propositions 2.1 and 2.2 can be seen as the generalizations of the results of avre and Skogestad [58] to SISO systems with multiple unstable poles and time delay.

Remark 2.2 The time-delay enters (2.41)-(2.42) assuming the form $e^{\theta p_i}$ and thus does not pose any serious limitations on input performance for systems with slow instabilities and *vice versa*. It follows from Corollary 2.1 that time delay essentially reduces the controllability (or observability) of poles and the faster the instability, the weaker the controllability (or observability) of the pole is, as compared to the delay-free system.

2.5 MIMO systems

In this section, we generalize the results of the previous section to MIMO systems. It is shown that the achievable input performance primarily depends on the joint controllability and observability of unstable poles of the system. These results can be directly used for selection of the subset of controlled and manipulated variables for stabilization.

2.5.1 Rational Systems

Similar to SISO systems, the achievable input performance is first characterized for rational systems. These results are extended to MIMO systems with time delay later in this section. To obtain expressions for I_2 and I_∞ for MIMO systems, we relate $\mathbf{X}$ and $\mathbf{Y}$ solving the AREs (2.9)-(2.10) to the Hankel singular values of $\mathcal{U}(\hat{\mathbf{G}}(s))^*$. When $\hat{\mathbf{G}}(s)$ has distinct unstable poles, the next lemma also provides an alternate expression for the Hankel singular values of $\mathcal{U}(\hat{\mathbf{G}}(s))^*$, which can also be of independent interest.

Lemma 2.4 Let $\hat{\mathbf{G}}(s)$ be a rational system and $\mathbf{X}, \mathbf{Y} \succ 0$ solve the corresponding AREs (2.25)-(2.26). Then,

$$\sigma^2_{Hi}(\mathcal{U}(\hat{\mathbf{G}}(s))^*) = \lambda_i(\mathbf{X}^{-1}\mathbf{Y}^{-1}) \quad i = 1, \cdots n_p \tag{2.45}$$

Further, if $\mathbf{G}\hat{(}s)$ has distinct unstable poles, let $\mathcal{U}(\hat{\mathbf{G}}(s)) \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$, such that $\mathbf{P} = \mathrm{diag}(p_1 \cdots p_{n_p})$, $\mathrm{Re}(p_i) > 0$. Then $\sigma_{Hi}(\mathcal{U}(\hat{\mathbf{G}}(s))^*)$ is given as,

$$\sigma_{Hi}(\mathcal{U}(\hat{\mathbf{G}}(s))^*) = \lambda_i^{\frac{1}{2}}\left[\left((\mathbf{BB}^*) \circ \mathbf{M}\right)\left((\mathbf{C}^*\mathbf{C}) \circ \mathbf{M}\right)\right] \tag{2.46}$$

where $\mathcal{U}(\cdot)$ denotes the unstable part and $\mathbf{M}$ is defined by (2.31).

Proof: Pre- and post-multiplying (2.34) by $\mathbf{T}_1$ and $\mathbf{T}_1^*$ respectively, where $\mathbf{T}_1$ is a state transformation matrix,

$$\begin{aligned} \mathbf{T}_1\mathbf{P}\mathbf{X}^{-1}\mathbf{T}_1^* + \mathbf{T}_1\mathbf{X}^{-1}\mathbf{P}^*\mathbf{T}_1^* &= \mathbf{T}_1\mathbf{B}\mathbf{B}^*\mathbf{T}_1^* \\ \Leftrightarrow \bar{\mathbf{P}}\bar{\mathbf{X}}^{-1} + \bar{\mathbf{X}}^{-1}\bar{\mathbf{P}}^* &= \bar{\mathbf{B}}\bar{\mathbf{B}}^* \end{aligned} \tag{2.47}$$

where $\bar{\mathbf{P}} = \mathbf{T}_1\mathbf{P}\mathbf{T}_1^{-1}$, $\bar{\mathbf{B}} = \mathbf{T}_1\mathbf{B}$ and $\bar{\mathbf{X}} = \mathbf{T}_1^{-*}\mathbf{X}\mathbf{T}_1^{-1}$. Similarly, by setting $\bar{\mathbf{C}} = \mathbf{C}\mathbf{T}_1^{-1}$ and $\bar{\mathbf{Y}} = \mathbf{T}_1\mathbf{Y}\mathbf{T}_1^*$,

$$\bar{\mathbf{P}}^*\bar{\mathbf{Y}}^{-1} + \bar{\mathbf{Y}}^{-1}\bar{\mathbf{P}} = \bar{\mathbf{C}}^*\bar{\mathbf{C}} \tag{2.48}$$

Now $\bar{\mathbf{Y}}^{-1}$ and $\bar{\mathbf{X}}^{-1}$ are the controllability and observability gramians of the stable system $\mathcal{U}(\hat{\mathbf{G}}(s))^* \leftrightarrow (-\bar{\mathbf{P}}^*, \bar{\mathbf{C}}^*, \bar{\mathbf{B}}^*)$ and (2.47)-(2.48) are the corresponding Lyapunov equations. If $\mathbf{T}_1$ is chosen such that $(-\bar{\mathbf{P}}^*, \bar{\mathbf{C}}^*, \bar{\mathbf{B}}^*)$ is a balanced realization, then $\bar{\mathbf{X}}^{-1} = \bar{\mathbf{Y}}^{-1} = \mathrm{diag}(\sigma_{Hi}(\mathcal{U}(\hat{\mathbf{G}}(s))^*))$ [117] and

$$\sigma_{Hi}^2(\mathcal{U}(\hat{\mathbf{G}}(s))^*) = \lambda_i(\bar{\mathbf{X}}^{-1}\bar{\mathbf{Y}}^{-1}) = \lambda_i(\mathbf{T}_1^{-*}\mathbf{X}^{-1}\mathbf{Y}^{-1}\mathbf{T}_1^*) = \lambda_i(\mathbf{X}^{-1}\mathbf{Y}^{-1}).$$

When $\hat{\mathbf{G}}(s)$ has distinct unstable poles, the alternate expression for the Hankel singular values of $\mathcal{U}(\hat{\mathbf{G}}(s))^*$ can be obtained by substituting for $\mathbf{X}^{-1}$ and $\mathbf{Y}^{-1}$ in (2.45) using Lemma 2.1. ■

Proposition 2.3 For the rational MIMO system $\hat{\mathbf{G}}(s)$ having n_p unstable poles, let $(-\bar{\mathbf{P}}^*, \bar{\mathbf{C}}^*, \bar{\mathbf{B}}^*)$ be the balanced realization of $\mathcal{U}(\hat{\mathbf{G}}(s))^*$. Then

$$I_2^2 = \sum_{i=1}^{n_p} \frac{2|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|}{\sigma_{Hi}^2(\mathcal{U}(\hat{\mathbf{G}}(s))^*)} \tag{2.49}$$

$$I_\infty = \underline{\sigma}_H^{-1}(\mathcal{U}(\hat{\mathbf{G}}(s))^*) \tag{2.50}$$

Proof: (1) For (2.49), based on the expression for I_2^2 (2.29),

$$I_2^2 = \mathrm{tr}(\mathbf{B}^*\mathbf{X}\mathbf{Y}\mathbf{X}\mathbf{B}) = \mathrm{tr}(\bar{\mathbf{B}}^*\bar{\mathbf{X}}\bar{\mathbf{Y}}\bar{\mathbf{X}}\bar{\mathbf{B}}) = \mathrm{tr}(\bar{\mathbf{B}}\bar{\mathbf{B}}^*\bar{\mathbf{X}}\bar{\mathbf{Y}}\bar{\mathbf{X}})$$

Define $\mathbf{\Sigma}_H = \mathrm{diag}(\sigma_{Hi}(\mathcal{U}(\hat{\mathbf{G}}(s))^*))$. Since $(-\bar{\mathbf{P}}^*, \bar{\mathbf{C}}^*, \bar{\mathbf{B}}^*)$ is the balanced realization of $\mathcal{U}(\hat{\mathbf{G}}(s))^*$, using Lemma 2.4 and setting $\bar{\mathbf{X}} = \bar{\mathbf{Y}} = \mathbf{\Sigma}_H^{-1}$,

$$\begin{aligned} I_2^2 &= \mathrm{tr}\left[(-\bar{\mathbf{P}}\mathbf{\Sigma}_H - \mathbf{\Sigma}_H\bar{\mathbf{P}}^*)\mathbf{\Sigma}_H^{-3}\right] \\ &= \mathrm{tr}(-\bar{\mathbf{P}}\mathbf{\Sigma}_H^{-2}) + \mathrm{tr}(-\mathbf{\Sigma}_H^{-2}\bar{\mathbf{P}}^*) = \sum_{i=1}^{n_p} \frac{|\bar{\mathbf{P}}_{ii} + \bar{\mathbf{P}}_{ii}^*|}{\sigma_{Hi}^2(\mathcal{U}(\hat{\mathbf{G}}(s))^*)} \end{aligned}$$

where $|\bar{\mathbf{P}}_{ii} + \bar{\mathbf{P}}_{ii}^*| = 2|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|$.

(2) For (2.50), based on the expression for I_∞ (2.30) and Lemma 2.4

$$I_\infty \;=\; \underline{\lambda}^{-\frac{1}{2}}(\mathbf{X}^{-1}\mathbf{Y}^{-1}) = \underline{\sigma}_H^{-1}(\mathcal{U}(\hat{\mathbf{G}}(s))^*)$$

■

The expressions (2.49)-(2.50) show that I_2 and I_∞ mainly depend on $\sigma_{Hi}(\mathcal{U}(\hat{\mathbf{G}}(s))^*)$, which is a measure of joint controllability and observability of the unstable poles.

Glover [43] studied the robust stability of systems in the presence of additive unstructured uncertainty. With the additive description of uncertainty, maximizing robust stability is equivalent to minimizing the $\mathcal{H}_\infty$ norm of transfer matrix from disturbances to inputs. Thus, the results of Glover [43] are also applicable to the present case of minimization of input energy required for stabilization. The expression for I_∞ as derived here is as an alternative proof of the similar result of Glover [43], but is generalized to the case where $\mathbf{W}_u(s)$ and $\mathbf{G}_w(s)$ can be minimum phase and share common unstable poles with the system.

Remark 2.3 In general, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of a transfer matrix can be arbitrarily apart. Proposition 2.3 shows that when input norm is minimized, I_2/I_∞ is always bounded as

$$2\frac{\underline{\sigma}_H^2(\mathcal{U}(\hat{\mathbf{G}}(s))^*)}{\bar{\sigma}_H^2(\mathcal{U}(\hat{\mathbf{G}}(s))^*)}\sum_{i=1}^{n_p}|\mathrm{Re}(\bar{\mathbf{P}}_{ii})| \le \frac{I_2^2}{I_\infty^2} \le 2\sum_{i=1}^{n_p}|\mathrm{Re}(\bar{\mathbf{P}}_{ii})| \tag{2.51}$$

where $\bar{\mathbf{P}}$ is the state matrix of the balanced realization of $\mathcal{U}(\hat{\mathbf{G}}(s))$. The closeness of I_2 and I_∞ follows from the fact that the related AREs (2.25)-(2.26) for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cases are the same. The ratio $\kappa_H = \bar{\sigma}_H(\mathcal{U}(\hat{\mathbf{G}}(s))^*)/\underline{\sigma}_H(\mathcal{U}(\hat{\mathbf{G}}(s))^*)$ is the condition number of $\mathcal{U}(\hat{\mathbf{G}}(s))^*$ expressed in terms of Hankel singular values and can be interpreted similar to the Euclidian condition number. A system that has a large Euclidian condition number has strong directionality and may be difficult to control [102, p.87]. Similarly, κ_H can be large due to small $\underline{\sigma}_H(\mathcal{U}(\hat{\mathbf{G}}(s))^*)$ indicating that the input requirement for stabilization is large. When $\kappa_H = 1$, the upper and lower bounds on I_2^2/I_∞^2 in (2.51) are the same with $I_2^2/I_\infty^2 = 2\sum_{i=1}^{n_p}|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|$.

In this chapter, we assumed that the disturbances enter the closed loop system through output channels. Proposition 2.3 can easily be applied to cases, where disturbances enters through input channels by setting $\mathbf{G}_w(s) = \mathbf{G}(s)$ (see Figure 2.5). For minimum phase systems affected by input disturbances, the expressions for achievable input performance are much simplified, as earlier shown by Chen *et al.* [26]. The results of Chen *et al.* [26] are shown to be a special case of Proposition 2.3 by the next Corollary.

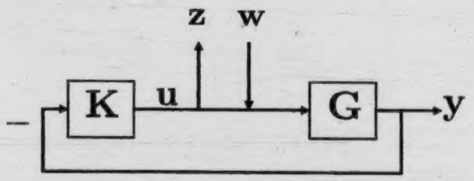

Figure 2.5: Disturbances entering through input channels

Corollary 2.3 With reference to Figure 2.5, let $\mathbf{G}(s)$ be minimum phase, right invertible and has n_p unstable poles. Then,

$$I_2^2 = 2\sum_{i=1}^{n_p} \mathrm{Re}(p_i); \qquad I_\infty = 1 \tag{2.52}$$

Proof: Let $\mathbf{G}(s) = \mathbf{G}_s\mathcal{B}_{po}^{-1}(s)$ such that $\mathbf{G}_s(s)$ is stable. With $\mathbf{G}_w(s) = \mathbf{G}(s)$ and using (2.17),

$$\begin{aligned} \|\mathbf{T}_{uw}(s)\| &= \|(\mathbf{I} + \mathbf{K}\mathbf{G}_s\mathcal{B}_{po}^{-1}(s))^{-1}\mathbf{K}\mathbf{G}_s\mathcal{B}_{po}^{-1}(s))\| \\ &= \|(\mathbf{I} + \hat{\mathbf{K}}\mathcal{B}_{po}^{-1}(s))^{-1}\hat{\mathbf{K}}(s)\| \end{aligned}$$

where $\hat{\mathbf{K}}(s) = \mathbf{K}\mathbf{G}_s(s)$. Let $(\bar{\mathbf{P}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}})$ be the balanced realization of $\mathcal{B}_{po}^{-*}(s)$. Since $\mathcal{B}_{po}^{-*}(s)$ is all-pass and stable, $\sigma_{Hi}(\mathcal{B}_{po}^{-*}(s)) = 1$ [42]. Then, using Proposition 2.3, $I_\infty = 1$ and $I_2^2 = \sum_{i=1}^{n_p} 2|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|$. The expression for I_2 follows by noting that $\bar{\mathbf{P}}_{ii} = p_i$ (*cf.* (2.16)). ■

The achievable input performance for multivariate systems depends on pole locations as well as pole directions. To illustrate this, we consider two extreme cases: (1) all the pole directions are orthogonal and (2) are co-linear with each other.

Corollary 2.4 Let $\mathbf{G}_1(s)$ and $\mathbf{G}_2(s)$ be rational systems with distinct unstable poles, where $\mathcal{U}(\mathbf{G}_1(s)) \leftrightarrow (\mathbf{P}, \mathbf{B}_1, \mathbf{C}_1)$, $\mathcal{U}(\mathbf{G}_2(s)) \leftrightarrow (\mathbf{P}, \mathbf{B}_2, \mathbf{C}_2)$ such that $\mathbf{P} = \mathrm{diag}(p_1 \cdots p_{n_p})$, $\mathrm{Re}(p_i) > 0$. Let $\|[\mathbf{B}_1']_i\|_2 = \|[\mathbf{B}_2']_i\|_2$, $\|[\mathbf{C}_1]_i\|_2 = \|[\mathbf{C}_2]_i\|_2$ for all $i = 1 \cdots n_p$ and

$$\begin{array}{lrcl} & \mathbf{y}_{pi}^*(\mathbf{G}_1(s))\mathbf{y}_{pj}(\mathbf{G}_1(s)) = 1 & \text{and} & \mathbf{u}_{pi}^*(\mathbf{G}_1(s))\mathbf{u}_{pj}(\mathbf{G}_1(s)) = 1 \quad \forall i, j \\ \text{and} & \mathbf{y}_{pi}^*(\mathbf{G}_2(s))\mathbf{y}_{pj}(\mathbf{G}_2(s)) = 0 & \text{and} & \mathbf{u}_{pi}^*(\mathbf{G}_2(s))\mathbf{u}_{pj}(\mathbf{G}_2(s)) = 0 \quad \forall i \neq j \end{array}$$

Then, $I_\infty(\mathbf{G}_1(s)) \geq I_\infty(\mathbf{G}_2(s))$.

Proof: Define the diagonal matrices $\mathbf{D}_I = \mathrm{diag}(\|[\mathbf{B}_1']_i\|_2) = \mathrm{diag}(\|[\mathbf{B}_2']_i\|_2)$ and $\mathbf{D}_O = \mathrm{diag}(\|[\mathbf{C}_1]_i\|_2) = \mathrm{diag}(\|[\mathbf{C}_2]_i\|_2)$. Based on the alternate expression for Hankel singular values (2.46) and $\mathbf{M}$ (2.31),

$$\underline{\sigma}_H(\mathcal{U}(\mathbf{G}_2(s))^*) = \underline{\lambda}^{\frac{1}{2}}\left[((\mathbf{D}_I[\mathbf{u}_{pi}^*(\mathbf{G}_2(s))\mathbf{u}_{pj}(\mathbf{G}_2(s))]_{ij}\mathbf{D}_I)\circ\mathbf{M})\right.$$
$$\left.((\mathbf{D}_O[\mathbf{y}_{pi}^*(\mathbf{G}_2(s))\mathbf{y}_{pj}(\mathbf{G}_2(s))]_{ij}\mathbf{D}_O)\circ\mathbf{M})\right]$$

Since $\mathbf{u}_{pi}^*(\mathbf{G}_2(s))\mathbf{u}_{pj}(\mathbf{G}_2(s)) = 0$ for all $i \neq j$,

$$\begin{aligned}
\underline{\sigma}_H(\mathcal{U}(\mathbf{G}_2(s))^*) &= \underline{\lambda}^{\frac{1}{2}}\left[(\mathbf{D}_I^2\circ\mathbf{M})(\mathbf{D}_O^2\circ\mathbf{M})\right] \\
&= \underline{\lambda}^{\frac{1}{2}}\left[\mathbf{D}_I\mathrm{diag}(1/(p_i+p_i^*))\mathbf{D}_I\mathbf{D}_O\mathrm{diag}(1/(p_i+p_i^*))\mathbf{D}_O\right] \\
&= \underline{\sigma}\left(\mathbf{D}_I\mathrm{diag}(1/(p_i+p_i^*))\mathbf{D}_O\right) = \min_i\,[\mathbf{D}_I^{-1}\mathbf{D}_O^{-1}]_{ii}(p_i+p_i^*)^{-1}
\end{aligned}$$

Similarly, it can be shown that, $\underline{\sigma}_H(\mathcal{U}(\mathbf{G}_1(s))^*) = \underline{\sigma}\left(\mathbf{D}_I\mathbf{M}\mathbf{D}_O\right)$. Using Proposition 2.3, $I_\infty(\mathbf{G}_1) = \bar{\sigma}\left(\mathbf{D}_I^{-1}\mathbf{M}^{-1}\mathbf{D}_O^{-1}\right)$ and using Lemma 2.2,

$$\begin{aligned}
I_\infty(\mathbf{G}_1) &\geq \max_i[\mathbf{D}_I^{-1}\mathbf{M}^{-1}\mathbf{D}_O^{-1}]_{ii} \\
&\geq \max_i[\mathbf{D}_I^{-1}\mathbf{D}_O^{-1}]_{ii}(p_i^*+p_i)\prod_{\substack{k=1\\k\neq i}}^{n_p}\frac{(p_i^*+p_k)^2}{(p_i^*-p_k^*)^2} \\
&\geq \max_i\,[\mathbf{D}_I^{-1}\mathbf{D}_O^{-1}]_{ii}(p_i^*+p_i) = I_\infty(\mathbf{G}_2)
\end{aligned}$$

where the first inequality holds since the maximum singular value of a matrix is always greater than or equal to the individual elements of the matrix. ■

In Corollary 2.4, the lengths of the pole vectors are assumed equal to highlight the effect of angles between the pole directions. In general, the optimal orientation of pole directions for input performance depends on the unstable pole locations and the Euclidian length of pole vectors. Intuitively, the input requirement for stabilization is minimized if pole directions are oriented such that the fastest instability is affected most and so on.

Example 2.2 Consider the following system,

$$\mathbf{G}(s) = \left[\begin{array}{cc|cc} 1 & 0 & \cos(\beta) & \sin(\beta) \\ 0 & 2 & \sin(\beta) & \cos(\beta) \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]; \quad p_1, p_2 \in \mathbb{R}, p_1 < p_2$$

For this system, $u_{p,1}^*u_{p,2} = \sin(2\beta)$. The variation of I_∞ with β is shown in Figure 2.6. The input requirement is maximum, when the pole directions are co-linear ($\beta = 0°$) and and is approximately 4 times larger than the case, where the pole directions are orthogonal ($\beta = 45°$). An explanation of this observation is as follows: When the pole directions are

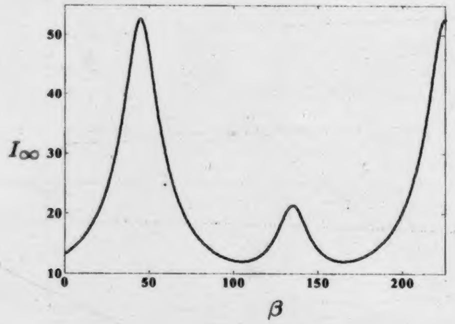

Figure 2.6: Effect of pole directions on I_∞

co-linear, the $\mathbf{B}$ matrix of the state space realization of $\mathbf{G}(s)$ is singular. The inputs affect the poles only after being filtered through the singular $\mathbf{B}$ matrix. Though the input itself can vary in all directions, when filtered through $\mathbf{B}$, it is effective only in a few directions increasing the input usage for stabilization.

2.5.2 Time Delay Systems

For extending Proposition 2.2 to MIMO systems, we use a similar method as used for univariate systems, *i.e.* by using a rational approximation of the time delay system and then letting the order of approximation approach infinity. We consider systems that can be expressed as

$$\hat{\mathbf{G}}(s) = \tilde{\mathbf{G}}(s) \circ \mathbf{\Theta}(s);\ \mathbf{\Theta}(s) = \left[e^{-\theta_{ij}s}\right] \tag{2.53}$$

where $\tilde{\mathbf{G}}$ is the delay-free part of the system. A system such as $\hat{\mathbf{G}}(s)$ in (2.53) with delay associated with individual elements of the transfer matrix, which cannot be separated at inputs or outputs, is sometimes referred to as a multiple delay system in the literature. It is pointed out that (2.53) does not represent the most general case and in practice is satisfied only when the $\mathbf{W}_u(s)$ and $\mathbf{G}_w(s)$ are diagonal. The remaining discussion in this section is limited to the cases where $n_y \geq n_u$ and similar expressions for $n_y < n_u$ can be obtained with minor modifications.

Lemma 2.5 Consider $\mathbf{H}(s) \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$ such that $\mathbf{P} = \text{diag}(p_1 \cdots p_{n_p})$, $\text{Re}(p_i) > 0$. Let $\mathbf{H}_1(\mathbf{s}) \in \mathcal{RH}_\infty$ with no zeros at p_i. Then

$$\mathcal{U}(\mathbf{H}_1(s) \circ \mathbf{H}(s)) = \sum_{i=1}^{n_p} \frac{1}{s - p_i} \mathbf{H}_1(p_i) \circ (\mathbf{C}_i \mathbf{B}_i') \tag{2.54}$$

The proof of Lemma 2.5 is similar to the proof of Lemma 2.3 and is omitted. We make the following additional assumption:

Assumption 2.3 Let $\mathcal{U}(\tilde{\mathbf{G}}(s)) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$. Then the matrix $(\tilde{\mathbf{C}}_i \tilde{\mathbf{B}}_i') \circ \mathbf{\Theta}(p_i)$ has full column rank for all $i = 1 \cdots n_p$.

Proposition 2.4 Consider that the MIMO system expressed by (2.53) has distinct poles and the system satisfies Assumption 2.3. Let $\mathcal{U}(\tilde{\mathbf{G}}) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ such that $\mathbf{P} = \text{diag}(p_1 \cdots p_{n_p})$, $\text{Re}(p_i) > 0$. If $\mathbf{G}_p \leftrightarrow (\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p)$, where

$$\begin{aligned} \mathbf{A}_p &= \text{diag}(p_1 \mathbf{I}_{n_u} \cdots p_{n_p} \mathbf{I}_{n_u}); \quad \mathbf{B}_p = [\mathbf{I}_{n_u} \cdots \mathbf{I}_{n_u}]' \\ \mathbf{C}_p &= \left[(\tilde{\mathbf{C}}_1 \tilde{\mathbf{B}}_1') \circ \mathbf{\Theta}(p_1) \cdots (\tilde{\mathbf{C}}_{n_p} \tilde{\mathbf{B}}_{n_p}') \circ \mathbf{\Theta}(p_{n_p}) \right] \end{aligned}$$

Then, $I_2(\hat{\mathbf{G}}) = I_2(\mathbf{G}_p)$, $I_\infty(\hat{\mathbf{G}}) = I_\infty(\mathbf{G}_p)$.

Proof: Let $\Theta(s)$ be approximated by an n^{th} order rational function as before. As $n \to \infty$, using Lemma 2.5 and the same arguments as used in the proof of Proposition 2.2,

$$\mathcal{U}(\hat{\mathbf{G}}(s)) = \sum_{i=1}^{n_p} \frac{1}{s - p_i} (\tilde{\mathbf{C}}_i \tilde{\mathbf{B}}_i') \circ \mathbf{\Theta}(p_i) \tag{2.55}$$

Due to Assumption 2.3, $\frac{1}{s-p_i} \mathbf{\Theta}(p_i) \circ (\mathbf{C}_i \mathbf{B}_i') \leftrightarrow (p_i \mathbf{I}_{n_u}, \mathbf{I}_{n_u}, \mathbf{\Theta}(p_i) \circ (\mathbf{C}_i \mathbf{B}_i'))$. Then the result follows by considering the aggregation of these subsystems. ■

It is interesting to note that when $\mathbf{\Theta}(s)$ is unstructured (delays cannot be separated at inputs or outputs), stabilization of the irrational system with n_p unstable poles is equivalent to stabilizing a rational system with $n_p \times n_u$ unstable poles. For systems not satisfying Assumption 2.3, the triplet $(\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p)$ is not necessarily a minimal realization. This assumption can be relaxed for generalization purposes, but this makes the expressions difficult and complex. A practical case, where Assumption 2.3 is always violated, occurs when the delays are associated with the sensors or actuators of the system. Systems with delay associated with sensors are handled next and the expressions for systems with delay associated with actuators can be obtained analogously.

Corollary 2.5 Let $\hat{\mathbf{G}}(s) = \text{diag}(e^{-\theta_i s}) \tilde{\mathbf{G}}(s)$ and $\mathcal{U}(\tilde{\mathbf{G}}(s)) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ such that $\mathbf{P} = \text{diag}(p_1 \cdots p_{n_p})$, $\text{Re}(p_i) > 0$, $p_i \neq p_j$. Let $\mathbf{G}_p(s) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{B}}, \mathbf{C}_p)$, where

$$\mathbf{C}_p = \left[\ \text{diag}(e^{-\theta_i p_1}) \tilde{\mathbf{C}}_1 \ \ \cdots \ \ \text{diag}(e^{-\theta_i p_{n_p}}) \tilde{\mathbf{C}}_{n_p} \ \right]$$

Then, $I_2(\hat{\mathbf{G}}(s)) = I_2(\mathbf{G}_p(s))$ and $I_\infty(\hat{\mathbf{G}}(s)) = I_\infty(\mathbf{G}_p(s))$.

The proof of Corollary 2.5 follows by considering (2.55) and noting that $(\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i') \circ \mathbf{\Theta}(p_i) = \mathrm{diag}(e^{-\theta_i p_i})\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i'$. It was shown earlier that for SISO systems, I_2, I_∞ are non-increasing functions of θ, but this does not hold for MIMO systems.

Example 2.3 Consider the system $\mathbf{G}(s) = \tilde{\mathbf{G}}(s) \circ \mathbf{\Theta}(s)$, where

$$\tilde{\mathbf{G}}(s) = \left[\begin{array}{cc|cc} 0.2 & 0 & 2 & 3 \\ 0 & 0.5 & 1 & 4 \\ \hline 3 & 2 & 0 & 0 \\ 5 & 3 & 0 & 0 \end{array}\right]; \qquad \mathbf{\Theta}(s) = \begin{bmatrix} e^{-\alpha_1 s} & e^{-\alpha_2 s} \\ e^{-\alpha_2 s} & e^{-\alpha_1 s} \end{bmatrix}$$

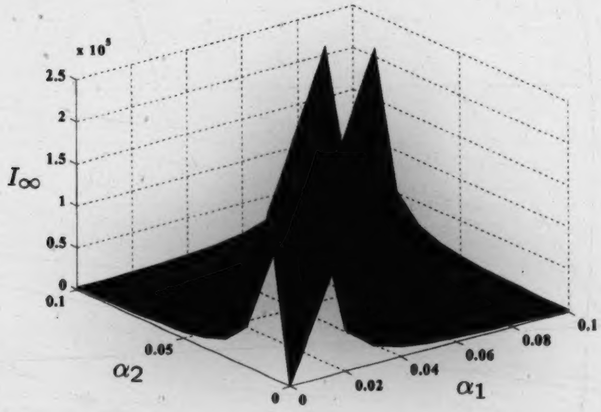

Figure 2.7: Variation of I_∞ with α_1 and α_2

The variation of I_∞ with α_1, α_2 is shown in Figure 2.7, which leads to the counter intuitive conclusion that the input requirement for stabilization for MIMO systems can decrease when the delay in some of the elements of the system increases. When $\alpha_1 \neq \alpha_2$, by virtue of Proposition 2.4, the unstable projection of the irrational system has 4 unstable poles (2 poles each at 0.2 and 0.5). However, when $\alpha_1 = \alpha_2 = \alpha$, $\mathbf{G}(s)$ can be expressed as $\mathbf{G}(s) = \tilde{\mathbf{G}}(s)e^{\alpha s}$. Then, using Corollary 2.5, the unstable projection of the irrational system has only 2 unstable poles. With slight abuse of terminology, the case of $\alpha_1 = \alpha_2 = \alpha$ can be interpreted as the system having 4 unstable poles and 2 unstable zeros at 0.2 and 0.5. Thus, when $\alpha_1 \neq \alpha_2$, these RHP zeros differ from their nominal values of 0.2 and 0.5 and effectively reduce the joint controllability and observability of the unstable poles. Keeping α_1 (or α_2) constant and increasing α_2 (or α_1), these RHP zeros recede away from the unstable poles reducing the input requirement for stabilization.

Corollary 2.6 Consider a MIMO system $\hat{\mathbf{G}}(s)$ that is expressed by (2.53) and satisfies Assumption 2.3. If $\hat{\mathbf{G}}(s)$ has a single real unstable pole p,

$$I_2^2 = \frac{8p^3}{\sum_{i=1}^{n_u} \sigma_i^2((\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \Theta(p))} \qquad I_\infty = \frac{2p}{\underline{\sigma}((\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \Theta(p))} \tag{2.56}$$

where $\mathcal{U}(\tilde{\mathbf{G}}(s)) \leftrightarrow (p, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$.

Proof: Define $\mathbf{G}_p(s) \leftrightarrow (p\mathbf{I}_{n_u}, \mathbf{I}_{n_u}, (\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \Theta(p))$. Now, similar to the proof of Proposition 2.4, it can be shown that $I_2(\hat{\mathbf{G}}(s)) = I_2(\mathbf{G}_p(s))$, $I_\infty(\hat{\mathbf{G}}(s)) = I_\infty(\mathbf{G}_p(s))$. Since $\mathbf{G}_p(s)$ has a single pole repeated n_u times, $\mathbf{M} = (1/2p)[\mathbf{1}_{n_u} \cdots \mathbf{1}_{n_u}]$. Using (2.46),

$$\begin{aligned} \sigma_{Hi}(\mathbf{G}_p(s)^*) &= (1/2p)\lambda_i^{1/2}\left[((\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \Theta(p))^*((\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \Theta(p))\right] \\ &= (1/2p)\sigma_i\left[(\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \Theta(p)\right] \end{aligned} \tag{2.57}$$

Now, (2.56) is obtained by substituting (2.57) in the expressions for I_2 and I_∞ (2.49)-(2.50).
■

For a system that is delay free and has a single unstable pole, $\mathbf{M} = 1/2p$, $\mathbf{B}\mathbf{B}^* = \|\mathbf{B}\|_2^2$ and $\mathbf{C}^*\mathbf{C} = \|\mathbf{C}\|_2^2$. Then, using the alternate expression for Hankel singular values (2.46),

$$I_2^2 = \frac{8p^3}{\|\mathbf{B}\|^2\|\mathbf{C}\|^2} \qquad I_\infty = \frac{2p}{\|\mathbf{B}\|\|\mathbf{C}\|} \tag{2.58}$$

This expression (2.58) was earlier obtained by Havre and Skogestad [58]. Propositions 2.3 and 2.4 can be seen as the generalization of the results of Havre and Skogestad [58] to systems with multiple unstable poles and time delay.

2.6 Extended Stability

The optimal controller that minimizes input requirement for stabilization cancels the stable poles of system (see § 2.3) and only unstable poles are moved. Though these stable poles do not appear in the closed loop transfer matrix from the disturbances to the inputs, they are still present in other closed loop transfer matrices, *e.g.* disturbances to outputs. When the system has lightly dampened stable poles, the variability of the output may be large. Further, when the linear model is obtained through linearization of an nonlinear system, the large variation of the lightly dampened modes can excite some nonlinearities. It is beneficial to *stabilize* the unstable as well as stable poles of the system that are close to the imaginary axis by moving them further into the left half of the complex plane. In the literature, this problem is known as the $\alpha-$stability problem, where all the modes of the closed loop system lie in a half plane satisfying $\mathrm{Re}(s) < -\alpha$ for the given positive scalar α (see [28] and reference within for details). In this section, we present a simple algorithm for the $\alpha-$stability problem with minimization of input usage.

For notational simplicity, we assume that $\mathbf{G}_w(s) = \mathbf{W}_u(s) = \mathbf{I}$. The algorithm is based on the following observation:

Observation 2.1 Let $[\mathbf{T}_{uw}(s)]_{\mathcal{H}_{2,opt}}$ and $[\mathbf{T}_{uw}(s)]_{\mathcal{H}_{\infty,sub}}$ represent the closed loop system from the disturbances to the inputs with the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal controllers implemented respectively. Then,

(1) The poles of $[\mathbf{T}_{uw}(s)]_{\mathcal{H}_{2,opt}}$ are given as the unstable poles of $\mathbf{G}(s)$ mirrored across the imaginary axis with multiplicity 2.

(2) A subset of poles of $[\mathbf{T}_{uw}(s)]_{\mathcal{H}_{\infty,sub}}$ are given as the unstable poles of $\mathbf{G}(s)$ mirrored across the imaginary axis.

Proof: (1) When only input performance is considered, the optimal controller cancels the stable part of the system (see § 2.3). Thus, we can consider the system as having only unstable poles without loss of generality. Let $\mathbf{G}(s) \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$, where $\mathrm{Re}(\lambda_i(\mathbf{P})) > 0$. Using the expression for optimal controller (2.7),

$$[\mathbf{T}_{uw}(s)]_{\mathcal{H}_{2,opt}} = \left[\begin{array}{cc|c} \mathbf{P} & \mathbf{BF} & \mathbf{0} \\ \mathbf{LC} & \mathbf{P}+\mathbf{BF}+\mathbf{LC} & -\mathbf{L} \\ \hline \mathbf{0} & \mathbf{F} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cc|c} \mathbf{P}+\mathbf{BF} & \mathbf{BF} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}+\mathbf{LC} & -\mathbf{L} \\ \hline \mathbf{F} & \mathbf{F} & \mathbf{0} \end{array}\right]$$

where the second equality is obtained by using a state transformation matrix $\mathbf{T}$ of the form,

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$$

Pre-multiplying the ARE (2.25) by $\mathbf{X}^{-1}$ and rearranging, $\mathbf{P} + \mathbf{BF} = -\mathbf{X}^{-1}\mathbf{P}^*\mathbf{X}$. Then, $\lambda_i(\mathbf{P} + \mathbf{BF}) = \lambda_i(-\mathbf{P}^*)$, $i = 1 \cdots n_p$. Similarly post-multiplying the ARE (2.26) by $\mathbf{Y}^{-1}$, $\lambda_i(\mathbf{P} + \mathbf{LC}) = \lambda_i(-\mathbf{P}^*)$, $i = 1 \cdots n_p$. The result follows by noting that the eigenvalues of $-\mathbf{P}^*$ are at the mirrored locations of the eigenvalues of $\mathbf{P}$.

(2) The proof is similar to the case of $[\mathbf{T}_{uw}(s)]_{\mathcal{H}_{2,opt}}$ and is omitted. ■

The fact that the controller minimizing input energy mirrors the unstable poles was earlier established by Kwakernaak and Sivan [76] for the LQG and by Glover [43] for the $\mathcal{H}_\infty$ optimal controller design problem. Kwakernaak and Sivan [76] justified this as a balance between the gain and decay rate of the inputs. Note that in the case of $\mathcal{H}_\infty$ optimal control, the remaining poles of $[\mathbf{T}_{uw}(s)]_{\mathcal{H}_{\infty,sub}}$ are given as $\lambda(\mathbf{P} + \mathbf{Z}_\infty\mathbf{LC})$, where as $\gamma^{-2} \rightarrow \rho(\mathbf{XY})^{-1}$, $\mathbf{Z}_\infty$ approaches singularity (*cf.* (2.11)). Thus, characterization of all the poles of $[\mathbf{T}_{uw}(s)]_{\mathcal{H}_{\infty,sub}}$ is difficult, but it does not have any effect on the algorithm for $\alpha-$stability, as presented next.

Algorithm 2.1 Consider that $\hat{\mathbf{G}}(s)$ is the generalized system (2.23). The $\alpha-$stability for this system can be achieved by following steps:

(a) Translate the imaginary axis by the transformation $s = \tilde{s} + \alpha/2$.

(b) Design an optimal controller for $\hat{\mathbf{G}}(\tilde{s})$, that minimizes the input requirement for stabilization.

(c) Use the inverse transformation $\tilde{s} = s - \alpha/2$ on $\mathbf{T}_{uw}(\tilde{s})$ to get a closed loop system that is $\alpha-$stable.

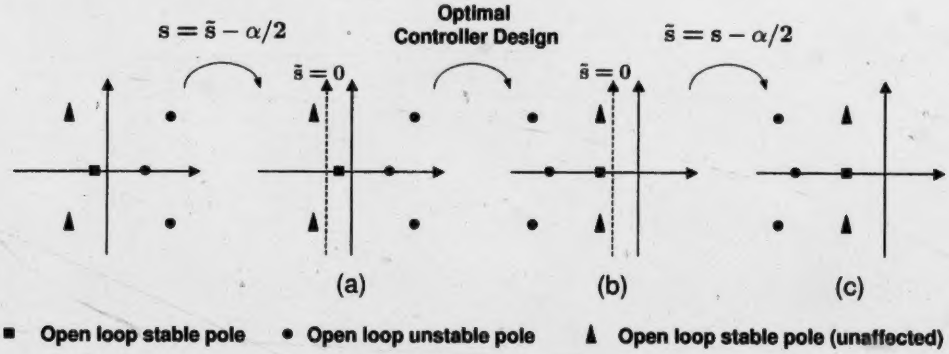
$\mathbf{s} = \tilde{\mathbf{s}} - \alpha/2$
Optimal
Controller Design
$\tilde{\mathbf{s}} = \mathbf{s} - \alpha/2$
$\tilde{\mathbf{s}} = 0$
$\tilde{\mathbf{s}} = 0$
(a)
(b)
(c)
■ Open loop stable pole
● Open loop unstable pole
▲ Open loop stable pole (unaffected)

Figure 2.8: Simple method for $\alpha-$stability

When the imaginary axis of the $s-$plane is translated to $\tilde{s} + \alpha/2$, the stable poles of the system that satisfy $\text{Re}(s) < -\alpha/2$ also appear in the RHP of the $\tilde{s}$-plane. The optimal controller that maximizes input performance reflects the poles in RHP of $\tilde{s}$-plane across the imaginary axis (see Observation 2.1) across the imaginary axis. Then, by inverse transformation to the $s-$plane, the poles of the closed loop system satisfy $\text{Re}(s) < -\alpha$. Using Proposition 2.3, the closed loop system satisfy

$$\begin{aligned} \|\mathbf{T}_{uw}(s-\alpha/2)\|_2^2 &= \sum_{i=1}^{n_p} \frac{2|\text{Re}(\bar{\mathbf{P}}_{ii})|}{\sigma_{Hi}^2(\mathcal{U}(\hat{\mathbf{G}}(s-\alpha/2))^*)} \\ \|\mathbf{T}_{uw}(s-\alpha/2)\|_\infty &= \underline{\sigma}_H^{-1}(\mathcal{U}(\hat{\mathbf{G}}(s-\alpha/2))^*) \end{aligned}$$

where $(\bar{\mathbf{P}}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$ is the balanced realization of $\hat{\mathbf{G}}(s-\alpha/2)$. Similar relations can be derived for a system with time delay using Proposition 2.4; however, expressing $\|\mathbf{T}_{uw}(s)\|$ directly in terms of $\|\hat{\mathbf{G}}(s)\|$ and α is difficult. This class of norms are called shifted norms and have been discussed by Boyd and Barratt [9, Ch. 5]. Nevertheless, Algorithm 2.1 provides a simplistic way of attaining $\alpha-$stability using available numerical tools for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal controller design.

2.7 Selection of Variables for Stabilizing Layer

The results presented earlier in this chapter are useful for selecting a subset of controlled and manipulated variables for stabilizing the system with minimum input usage. Clearly, the optimal set of variables can be selected by comparing the achievable input performance for different alternatives. A limitation of this approach is that it suffers from the curse of dimensionality, as the number of alternatives show an exponential growth with system dimensions. In this section, we present an iterative algorithm for finding a suboptimal solution in finite time.

Further, selection of variables for the stabilizing layer through minimizing input usage is beneficial, but generally there are also other criteria. For example, the effect of disturbances on the remaining control problem (see Figure 2.1) can be amplified due to closure of the stabilizing or inner loop making the task of performance satisfaction difficult. We show that this issue can also be addressed in the framework of input usage minimization.

2.7.1 Choice of Norm

For a rational system with a single unstable pole driven by pure measurement noise, the optimal subset of the controlled and manipulated variables is independent of the choice of norm [58]. In the general case, however, the choice of norm can influence the optimal combination of variables. For example, consider the following system,

$$\hat{\mathbf{G}}(s) = \frac{1}{(s-1)(s-2)} \begin{bmatrix} (0.7s-1.2) & -(2.2s+2.4) \end{bmatrix}$$

where the objective is to choose one of the inputs requiring minimum usage for stabilization. Use of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms suggests the selection of u_2 and u_1 respectively. The appropriate norm can be chosen based on the information available regarding the disturbance characteristics, *e.g.* when the disturbances can be considered to be white noise, use of $\mathcal{H}_2$ norm is appropriate. On the other hand, when only bounds on the disturbances are available, $\mathcal{H}_\infty$ norm should be used [115].

We note that the $\mathcal{L}_1$ norm closely addresses the physical constraints of the system and [117]

$$\|\hat{\mathbf{K}}(s)\hat{\mathbf{S}}(s)\|_\infty \leq \|\hat{\mathbf{K}}(s)\hat{\mathbf{S}}(s)\|_{\mathcal{L}_1}$$

Thus, use of $\mathcal{H}_\infty$ norm may be preferred over $\mathcal{H}_2$ norm. If for some combination of variables, $\|\hat{\mathbf{K}}(s)\hat{\mathbf{S}}(s)\|_\infty > \beta$, where β depends on physical constraints on the manipulated variables, system stabilization without actuator saturation using a linear feedback controller is not possible.

2.7.2 Reducing Computational Complexity

Consider a rational system with a single unstable pole, where the closed loop system is driven by measurement noise. For such systems, I_∞ and I_2 depend on $\|\mathbf{B}\|$ and $\|\mathbf{C}\|$ (*cf.* (2.58)) and the following conclusions can be drawn:

- The optimal set of m_y controlled and m_u manipulated variables can be found by selecting variables with largest entries in the **B** and **C** matrices.
- The optimal set of m_y controlled and m_u manipulated variables is always a subset of the optimal set of $(m_y + 1)$ controlled and $(m_u + 1)$ manipulated variables.

With this monotonic relationship, the optimal set of variables for stabilization can be selected through $(n_y + n_u)$ comparisons for a $n_y \times n_u$ dimensional system. Unfortunately, this attractive result does not hold for systems with multiple unstable poles. Specifically, consider that the set of controlled and manipulated variables be partitioned into subsets of equal dimensions as, $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2\ \mathbf{y}_3]$ and $\mathbf{u} = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$. Among these subsets, let the input requirement be minimized by choosing $\mathbf{y}_1, \mathbf{u}_1$. In general, there is no guarantee that the achievable input performance for the subset $[\mathbf{y}_1,\ \mathbf{y}_2], [\mathbf{u}_1\ \mathbf{u}_2]$ is better than the subset $[\mathbf{y}_2,\ \mathbf{y}_3], [\mathbf{u}_2\ \mathbf{u}_3]$. This point is further illustrated using the following system:

$$\mathbf{G}(s) = \frac{1}{(s-0.5)(s-1.7)} \left[\ (-1.7s + 0.75) \quad (-s + 1.1) \quad -0.3(s + 0.1)\ \right]$$

For this system, u_3 is the optimal choice for $m_u = 1$ and u_1, u_2 is the optimal choice for $m_u = 2$, when either of $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms are minimized. Due to the lack of a monotonic relationship, $C^{n_y}_{m_y} \times C^{n_u}_{m_u}$ comparisons are required for optimally selecting m_y controlled and m_u variables for a $n_y \times n_u$ dimensional system. Solving the variable selection problem through comparison of all alternatives is computationally intractable, as the number of alternatives grow exponentially with the system dimensions. To this end, Havre [56] has suggested using the following step-wise approach to obtain a suboptimal solution in finite time, where the unstable poles are stabilized one at a time:

Algorithm 2.2 For rational systems with $\mathbf{W}_u(s) = \mathbf{G}_w(s) = \mathbf{I}$,

(a) Scale the system variables and obtain a state space realization of the scaled system, where $\mathcal{U}(\mathbf{G}(s)) \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$ such that $\mathbf{P} = \text{diag}(p_1 \cdots p_{n_p})$, $\text{Re}(p_i) > 0$.

(b) To stabilize the first real or pair of complex unstable pole (preferably the fastest unstable pole), choose the controlled and manipulated variables with largest entries in the corresponding rows and columns of the $\mathbf{B}$ and $\mathbf{C}$ matrices respectively. Design a controller to stabilize the chosen unstable pole and close the loop.

(c) Similar to the previous steps, obtain a state space realization for the remaining control problem and stabilize the second unstable pole. Repeat the procedure until all unstable poles are stabilized.

This simple method avoids the problem of computational complexity, as in the worst case, maximum of $\sum_{i=0}^{n_p}(n_y + n_u - 2i)$ comparisons are required; however, it suffers from the following limitations:

- Algorithm 2.2 yields a decentralized controller designed sequentially and thus the input usage for stabilization is large as compared to a full block multivariate controller.
- In the worst case, this method requires that $n_r + n_c$ controlled and manipulated variables be used for a system with n_r real and n_c pairs of complex unstable poles.
- The algorithm does not handle time delay systems or the case where $\mathbf{W}_u(s) \neq \mathbf{I}$ or $\mathbf{G}_w(s) \neq \mathbf{I}$.

We next present an iterative method that does not suffer from the limitations of Algorithm 2.2. The central idea is to choose one controlled or manipulated variable at a time. The algorithm provides a reasonable suboptimal solution for the variable selection problem in finite time, where the computational time increases linearly with system dimensions and quadratically with the number of variables to be selected. The case of system stabilization using decentralized controller is handled in the next chapter.

Algorithm 2.3 Prior to variable selection, scale the system variables.

(a) Select the optimal set consisting of 1 controlled and 1 manipulated variable that minimizes input requirement for stabilization by enumerating all possible $n_y \times n_u$ alternatives.

(b) Keeping the set of chosen manipulated variables the same, select an additional controlled variable that minimizes input requirement for stabilization.

(c) Keeping the set of chosen controlled variables the same, select an additional manipulated variable that minimizes input requirement for stabilization.

(d) Repeat steps (b) and (c) until m_y controlled and m_u manipulated variables are selected. If $m_y \neq m_u$, skip step (b) or (c) once the required number of variables are selected.

Algorithm 2.3 can be easily used to handle time delay systems and the cases where $\mathbf{W}_u(s) \neq \mathbf{I}$ or $\mathbf{G}_w(s) \neq \mathbf{I}$. Note that when $\mathbf{W}_u(s)$ and $\mathbf{G}_w(s)$ are not diagonal, the algorithm requires inversion of different sub-matrices of $\mathbf{W}_u(s)$ and $\mathbf{G}_w(s)$ during every iteration. For selecting the set of m_y controlled and m_u manipulated variables for a $n_y \times n_u$ dimensional system, the Algorithm 2.3 requires $n_y n_u + \sum_{i=1}^{n_y - m_y + 1}(n_y - i) + \sum_{j=1}^{n_u - m_u + 1}(n_u - j)$ number of comparisons. This expression can be simplified as,

$$n_y n_u + (n_y - 0.5m_y)(m_y - 1) + (n_u - 0.5m_u)(m_u - 1)$$

Essentially, starting from the optimal set of 1 controlled and 1 manipulated variable, at every step, Algorithm 2.3 adds one locally optimal controlled or manipulated variable. A similar algorithm can be constructed that starts with all variables and eliminates one controlled or manipulated variable at every step. This alternative algorithm is particularly useful, when $m_y > n_y/2$ and $m_u > n_u/2$.

Example 2.4 We consider the base case of the Tennessee Eastman benchmark problem [33]. A linearized model of this process is obtained by numerical differentiation of the nonlinear model. The model is scaled prior to variable selection using the approach of Havre [56, Ch.6]. Based on the recommendation of Havre [56, Ch.6], we use only a subset of controlled variables and avoid using feed streams for stabilization. The resulting system has 11 controlled and 8 manipulated variables and unstable poles at $3.07 \pm j5.08$, $0.02 \pm j0.16$, 0.007 and 0.

In Table 2.1, we show the results obtained by applying Algorithm 2.3 for $\mathcal{H}_\infty$ norm minimization, which are compared against the optimal solution obtained by enumeration.

		Exact Solution			Suboptimal Solution		
m_y	m_u	CV	MV	I_∞	CV	MV	I_∞
1	1	y_{22}	u_{10}	0.11	y_{22}	u_{10}	0.11
1	2	y_{21}	u_8,u_{11}	0.077	y_{22}	u_{10}, u_{11}	0.1047
2	1	y_{12}, y_{21}	u_{10}	0.0235	y_{12}, y_{22}	u_{10}	0.084
2	2	y_{12}, y_{21}	u_{10},u_{11}	0.0222	y_{12}, y_{22}	u_{10}, u_{11}	0.0783
3	3	y_8, y_{12}, y_{21}	u_5, u_{10}, u_{11}	0.0212	y_{12}, y_{21}, y_{22}	u_5, u_{10}, u_{11}	0.0213

Table 2.1: Comparison of the results obtained using Algorithm 2.3 with the optimal solution for stabilization of Tennessee Eastman process using $\mathcal{H}_\infty$ optimal controller

m_y	m_u	CV	MV	I_2^2
1	1	y_{21}	u_{10}	0.0068
1	2	y_{21}	u_{10},u_{11}	0.0059
2	1	y_{12},y_{21}	u_{10}	0.0063
2	2	y_{11},y_{21}	u_{10},u_{11}	0.0055
3	3	y_{11},y_{12},y_{21}	u_5,u_{10},u_{11}	0.0050

Table 2.2: Alternatives for stabilizing Tennessee Eastman process using $\mathcal{H}_2$ optimal controller. Due to monotonicity, Algorithm 2.3 provides the optimal solution.

The suboptimal solution is reasonably close to the optimal solution, but is obtained using a fraction of the computational requirement for enumeration. For example, when $m_y = m_u = 3$, a total of 9240 comparisons are required for enumeration, where as Algorithm 2.3 requires only 120 comparisons.

For $\mathcal{H}_\infty$ norm minimization, the lack of the monotonic relationship should be noticed in Table 2.1. In particular, for $m_y = 1, m_u = 2$, choice of u_8, u_{11} is optimal, but this set does not contain u_{10}, which is optimal for $m_y = 1, m_u = 1$. On the contrary, when $\mathcal{H}_2$ norm is minimized, Algorithm 2.3 provides the same solution as obtained by enumeration. This happens as the optimal solution for $\mathcal{H}_2$ norm minimization shows monotonicity, but this is not true in general. The different alternatives for stabilization of the Tennessee Eastman process using an $\mathcal{H}_2$ optimal controller are shown in Table 2.2.

In general, m_y, m_u are not specified beforehand and are decided upon by trading them off against the achievable input performance. For this case study, the achievable $\mathcal{H}_\infty$ optimal input performance using all controlled and manipulated variables is 0.0194. Then, based on the optimal solution obtained by enumeration, use of 2 controlled and 1 manipulated variables is sufficient. In comparison, a disadvantage of Algorithm 2.3 is that is suggests use of 3 controlled and 3 manipulated variables and finding an improved algorithm remains an open area of research.

2.7.3 Other Criteria

In the previous section, the selection of variables for the stabilizing layer through minimizing input usage was demonstrated. Though beneficial, this approach can be insufficient for practical controller design problems as generally there are also other criteria. One such important criterion is the amplification of effect of disturbances on the remaining control problem (see Figure 2.1) due to closure of the stabilizing loop, which can make the task of performance satisfaction difficult. We show that this issue can also be addressed in the framework of input usage minimization.

Consider the set of controlled and manipulated variables be conformably partitioned as

$$\begin{aligned}\mathbf{y_1}(s) &= \mathbf{G}_{11}(s)\mathbf{u_1}(s) + \mathbf{G}_{12}(s)\mathbf{u_2}(s) + \mathbf{G}_{w1}(s)\mathbf{w}(s)\\ \mathbf{y_2}(s) &= \mathbf{G}_{21}(s)\mathbf{u_1}(s) + \mathbf{G}_{22}(s)\mathbf{u_2}(s) + \mathbf{G}_{w2}(s)\mathbf{w}(s)\end{aligned}$$

where the subset $\mathbf{y_2}, \mathbf{u_2}$ is used for stabilization. When the stabilizing loop is closed, the effect of the disturbance on the controlled variables of open loop system is given as [56],

$$\begin{aligned}\mathbf{y_1}(s) &= \left(\mathbf{G}_{w1}(s) - \mathbf{G}_{12}\mathbf{K}(s)(\mathbf{I} + \mathbf{G}_{22}\mathbf{K}(s))^{-1}\mathbf{G}_{w2}(s)\right)\mathbf{w}(s)\\ &= \mathbf{G}_{w1}(s)\left(\mathbf{I} - \mathbf{G}_{w1}^{-1}\mathbf{G}_{12}\mathbf{K}(s)(\mathbf{I} + \mathbf{G}_{22}\mathbf{K}(s))^{-1}\mathbf{G}_{w2}(s)\right)\mathbf{w}(s)\end{aligned}$$

Then, the stabilizing layer amplifies the effect of disturbances on the remaining control problem, if

$$\|\mathbf{I} - \mathbf{G}_{w1}^{-1}(s)\mathbf{G}_{12}\mathbf{K}(s)(\mathbf{I} + \mathbf{G}_{22}\mathbf{K}(s))^{-1}\mathbf{G}_{w2}(s)\|_\infty > 1$$

During the selection of controlled and manipulated variables, it is beneficial to take this effect of disturbance amplification into account. The stabilizing controller can be designed such that the input usage for stabilization is traded off against the disturbance amplification effect. For this purpose, we note that

$$\begin{aligned}&\|\mathbf{I} - \mathbf{G}_{w1}^{-1}(s)\mathbf{G}_{12}\mathbf{K}(s)(\mathbf{I} + \mathbf{G}_{22}\mathbf{K}(s))^{-1}\mathbf{G}_{w2}(s)\|_\infty\\ &\qquad\qquad \leq 1 + \|\mathbf{G}_{w1}^{-1}(s)\mathbf{G}_{12}\mathbf{K}(s)(\mathbf{I} + \mathbf{G}_{22}\mathbf{K}(s))^{-1}\mathbf{G}_{w2}(s)\|_\infty\end{aligned}$$

Thus, it suffices to minimize $\bar{\sigma}(\mathbf{G}_{w1}^{-1}(s)\mathbf{G}_{12}\mathbf{K}(s)(\mathbf{I} + \mathbf{G}_{22}\mathbf{K}(s))^{-1}\mathbf{G}_{w2}(s))$ at the desired frequencies. In general, we can minimize $\|\mathbf{W}_w(s)\mathbf{G}_{w1}^{-1}(s)\mathbf{G}_{12}\mathbf{K}(s)(\mathbf{I} + \mathbf{G}_{22}\mathbf{K}(s))^{-1}\mathbf{G}_{w2}(s)\|_\infty$, where $\mathbf{W}_w(s)$ is a frequency dependent weight. This requirement combined with minimization of input usage for stabilization results in a multi-objective optimization problem and a popular approach is to instead solve the following optimization problem

$$\min_{\mathbf{K}(s)} \left\| \begin{bmatrix} \mathbf{W}_u(s) \\ \mathbf{W}_w(s)\mathbf{G}_{w1}^{-1}(s)\mathbf{G}_{12} \end{bmatrix} \mathbf{K}(s)(\mathbf{I} + \mathbf{G}_{22}\mathbf{K}(s))^{-1}\mathbf{G}_{w2}(s) \right\|_\infty \tag{2.59}$$

This problem is the same as the general input usage minimization problem considered earlier in this chapter, except the special choice of frequency dependent weights. Thus, the controlled and manipulated variables can be selected as discussed in the previous section with minor modification. For $\mathcal{H}_2$ norm minimization, similar expression as (2.59) can be used.

2.8 Chapter Summary

In this chapter, we used a state space framework to obtain analytic expressions for achievable input performance for SISO and MIMO systems with and without time delay. Regarding the factors affecting achievable input performance, the following general conclusions are drawn:

1. The input performance primarily depends on the joint controllability and observability of unstable poles.
2. In the $\mathcal{H}_\infty$-control framework, there are no limitations on achievable input performance for minimum phase systems, when the closed loop system is driven by input disturbances. In the $\mathcal{H}_2$-control framework, the achievable input performance for this class of systems is limited only by the location and number of unstable poles.
3. Time delay poses no serious limitation on the achievable input performance for a system with slow instabilities and *vice versa*.
4. The input performance of a MIMO system, where the delays cannot be separated at the inputs or outputs, can be much worse as compared to a system with delays that can be factored at the inputs or outputs.

5. In contrast to SISO systems, the input requirement for stabilization may decrease for MIMO systems with an increase in time delay in some elements of the transfer matrix relating controlled and manipulated variables.

Based on the observation that the optimal controller mirrors the unstable poles across the imaginary axis, a simple method is proposed to handle the $\alpha-$stability problem. In the context of process control, this method is useful for controller design using available numerical tools for systems with a subset of poles on or near the imaginary axis frequently arising due to holdup of utilities and raw materials.

It is demonstrated that except for systems with a single unstable pole, the optimal subset of controlled and manipulated variables that minimizes input requirement for stabilization depends on the choice of norm. In the general case, the choice of norm depends on the available information regarding disturbance characteristics, but use of $\mathcal{H}_\infty$ norm can be preferred to address the actuator saturation issue. We also presented some insights to reduce the computational complexity of the variable selection problem and handle criteria other than input performance maximization in a unified framework.

2.9 Further Reading on Performance Limitations

The area of fundamental limitations of feedback control can be dated back to Bode [8]. In his seminal work, Bode showed that for stable systems with more than one pole-zero excess, the integral of logarithmic magnitude of the sensitivity function over all frequencies is always zero. With a finite bandwidth limitation, this result implies the unavoidable tradeoff between different performance objectives. Bode's result has been extended to open loop unstable systems by Freudenberg and Looze [38]. The same authors have also developed a Poisson-type integral to quantify the limitations imposed by the unstable zeros on the sensitivity integral. The classical Bode sensitivity and Poisson-type integrals have been extended to multivariate systems by Chen [21]. The importance of the Bode sensitivity integral for some real-life controller design problems is discussed by Stein [104].

Note that the Bode sensitivity or Poisson integrals always hold, irrespective of the optimal controller design criteria. A similar set of constraints, known as analyticity or interpolation constraints, were introduced by Zames [115]. The interpolation constraints show that for a system with unstable poles and zeros, peaks in sensitivity and complementary sensitivity functions are inevitable [102, 115]. Havre and Skogestad [57] have used these interpolation constraints to quantify limitations imposed by RHP poles and zeros on the lower bounds on several important closed loop transfer matrices. Chen [22] has presented improved achievable bounds on the sensitivity and complementary sensitivity functions through the use of these constraints combined with Nevanlinna-Pick interpolation theory [6].

Over the years, a number of results have been obtained for a related class of problems, where the achievable performance is quantified assuming a particular performance criterion. One of the most studied problems is the singular or cheap control problem. For discrete time systems, Peng and Kinnaert [89] have provided an explicit solution and the achievable performance is characterized by Qiu and Davison [92]. Recently, Yuz and Goodwin [114] presented an approximate solution to the decentralized minimum variance control, which is also studied later in this thesis. The presented list of references on performance limitations is far from complete. The reader is encouraged to refer to the books [9, 97, 102] and the recently published special issue on performance limitations by IEEE Transactions on Automatic Control [27].

Chapter 3

μ-Interaction Measure for Unstable Systems

The requirement that the block diagonal part of the system should have the same unstable poles as the system limits the practical applicability of conventional μ-interaction measure (μ-IM) [49] to stable systems. This limitation can be overcome by designing the decentralized controller based on a block diagonal approximation that is different from the block diagonal elements, but has the same number of unstable poles as the system. By expressing the μ-IM in terms of the transfer matrix between the disturbances and inputs, we show that the block diagonal approximation can be sub-optimally selected by minimizing the scaled $\mathcal{L}_\infty$ distance between the system and the approximation. We present a numerical method for choosing the block diagonal approximation and a simple method for designing the decentralized controller based on the approximation.[1]

3.1 Introduction

The last chapter presented results on system stabilization using minimal control action. In this chapter, we consider the system stabilization using a decentralized controller. Over the years, three different approaches have evolved for decentralized controller design:

a) *Simultaneous design using parametric search methods*: The decentralized controller is chosen to have a fixed structure (*e.g.* PID controller) with unknown parameters.

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[1] A part of this work was performed while the author was visiting Professor Sigurd Skogestad, Norwegian Institute of Science and Technology, Trondheim, Norway during March-May 2003.

The central idea of this chapter was presented at the American Control Conference, Boston, MA, 2004 [74].

The optimal value of these parameters is found by minimizing the appropriate norm of the closed loop system using direct or indirect search based methods. Though useful, this approach results in optimization problems that are not usually convex and can be highly complicated even for simple systems [7].

b) *Sequential design*: The controllers are designed sequentially using a lexicographical ordering of the individual controllers. The lowest level controller is designed first and the loop is closed. The next controller is designed based on the partially closed loop system. The resulting performance strongly depends on the ordering of the loops and often a trial and error approach is required to obtain acceptable performance [67, 84].

c) *Independent design*: The individual controllers are designed independently of each other based on a block diagonal approximation that is usually taken as the block diagonal elements of the system. Then, the decentralized controller design problem reduces to design of a number of small dimensional full multivariable controllers. When the interactions are small, such a controller also stabilizes the closed loop system with minimal loss of performance in comparison to the design basis [66, 101]. This approach always results in suboptimal performance because the tuning of other controllers is neglected.

In this work, we focus on the independent design approach. Although sub-optimal, the controller design is much simpler as compared to other techniques. Furthermore, this approach easily handles the cases in which only the bounds on (possibly time-varying) off-diagonal elements of the system are available [98].

Grosdidier and Morari [49] proposed the use of μ interaction measure (μ-IM) to assess the feasibility of system stabilization through independent designs of individual loops. This approach yields sufficient conditions to ensure that the decentralized controller that stabilizes the block diagonal part of the system also stabilizes the system itself. The problem of decentralized controller synthesis through independent designs has also been studied by Limbeer [78] and Ohta *et al.* [88], who used the concepts of generalized block diagonal dominance and quasi block diagonal dominance respectively. The use of $\mu-$IM is less conservative than these approaches because the controller structure is taken into account. A connection between these methods based on dominance and $\mu-$IM is established in the next chapter.

The conventional μ-IM requires that the system and its block diagonal part have the same right half plane (RHP) poles. Grosdidier and Morari [49] pointed out that this condition is not satisfied by most of the systems encountered in practice, limiting the applicability of

μ-IM to open loop stable systems. Samyudia *et al.* [96] have criticized the μ-IM for this limitation and have instead proposed a method based on ν-gap metric [110]. In this chapter, we present a modified μ-IM that easily handles unstable systems. The decentralized controller is designed based on a block diagonal approximation that is different from the block diagonal elements, but has the same number of unstable poles as the system.

Clearly, the number of block diagonal systems with the required number of unstable poles is infinite and the success of the modified μ-IM approach strongly depends on the choice of an appropriate approximation. We express the μ-IM in terms of the closed loop transfer matrix between disturbances and system input (or controller output). This alternate representation shows that the block diagonal approximation can be reasonably selected by minimizing the scaled $\mathcal{L}_\infty$ distance between the system and the approximation. The problem of finding a structured approximation of a full multivariate system has earlier been considered by Li and Zhou [77], but no numerical methods for solving the approximation problem are provided. In this chapter, we present a numerical approach, where the approximation problem is first solved at a set of chosen frequencies followed by a parametric identification method.

Similar to the conventional μ-IM method, the stabilizing decentralized controller can be synthesized using a loop shaping approach based on the block diagonal approximation. An advantage of alternate representation of μ-IM used here is that controller design can be much simplified using the results of last chapter. Although the focus of this chapter is on finding stabilizing decentralized controllers, we show that the stabilizing controller inherently minimizes an upper bound on the input requirement for stabilization. The results presented here can also be extended to handle (robust) performance issues directly using the results of Skogestad and Morari [101].

The organization of this chapter is as follows: some useful results from robust control theory and optimization are presented in § 3.2; the available results of μ-IM are reviewed and its limitation is pointed out in § 3.3; the alternate representation of μ-IM is presented and upper bounds on closed loop performance are derived in § 3.4; in § 3.5 we consider the problem of selecting the optimal block diagonal approximation; the simplified controller design is presented in § 3.6; in § 3.7, a numerical example is presented to demonstrate the utility of proposed approach followed by chapter summary in § 3.8.

3.2 Preliminaries

In this section, we review the useful concepts of structured singular value, model order reduction and optimization using linear matrix inequalities. These results are used extensively during the remaining development in this chapter.

3.2.1 Structured Singular Value

Most of the problems encountered in robust control theory can be reduced to guaranteeing that for some characteristic transfer matrix $\mathbf{M}(s)$, $\mathbf{I}-\mathbf{M}(j\omega)\boldsymbol{\Delta}(j\omega)$ remains nonsingular for all ω for all allowable values of $\boldsymbol{\Delta}(j\omega) \in \mathbb{C}^{q\times p}$. A practical and mathematically convenient way of representing the set of allowable $\boldsymbol{\Delta}(s)$ is as a norm bounded set, *e.g.*, $\bar{\sigma}(\boldsymbol{\Delta}(j\omega)) \leq 1$ for all ω. With this representation, the smallest norm of the destabilizing perturbation $\boldsymbol{\Delta}(j\omega)$ is given as $1/\bar{\sigma}(\mathbf{M}(j\omega))$. Then, it follows that $\det(\mathbf{I}-\mathbf{M}(j\omega)\boldsymbol{\Delta}(j\omega)) \neq 0$ for all allowable perturbations, iff $\|\mathbf{M}(s)\|_\infty < 1$.

In the above discussion, allowable perturbations include all matrices $\boldsymbol{\Delta}(j\omega)$ with $\bar{\sigma}(\boldsymbol{\Delta}(j\omega)) \leq 1$ for all ω. In practice, many problems arise, where $\boldsymbol{\Delta}(s)$ has a structure, *i.e.* some entries of $\boldsymbol{\Delta}(s)$ are identically zero. Then, the condition $\|\mathbf{M}(s)\|_\infty < 1$ is only sufficient (and highly restrictive) for ensuring that $\det(\mathbf{I}-\mathbf{M}(j\omega)\boldsymbol{\Delta}(j\omega)) \neq 0$ for all allowable perturbations. This motivates the use of the structured singular value, which explicitly accounts for the structure of the perturbations.

Definition 3.1 Let the set $\boldsymbol{\Delta} \in \mathbb{C}^{p\times q}$ be defined as

$$\boldsymbol{\Delta} = \{\mathrm{diag}(\boldsymbol{\Delta}_i) : \boldsymbol{\Delta}_i \in \mathbb{C}^{p_i\times q_i}, \bar{\sigma}(\boldsymbol{\Delta}) \leq 1\}$$

The structured singular value of $\mathbf{A} \in \mathbb{C}^{q\times p}$ is given as [35],

$$\mu_{\boldsymbol{\Delta}}(\mathbf{A}) = \frac{1}{\min\{\bar{\sigma}(\tilde{\boldsymbol{\Delta}}) : \tilde{\boldsymbol{\Delta}} \in \boldsymbol{\Delta}, \det(\mathbf{I}-\mathbf{A}\tilde{\boldsymbol{\Delta}}) = 0\}}$$

unless no $\tilde{\boldsymbol{\Delta}} \in \boldsymbol{\Delta}$ makes $(\mathbf{I}-\mathbf{A}\tilde{\boldsymbol{\Delta}})$ singular, in which case $\mu_{\boldsymbol{\Delta}}(\mathbf{A}) = 0$.

The $\mu_{\boldsymbol{\Delta}}(\mathbf{A})$ represents 2−norm of the smallest structured perturbation that makes $\mathbf{I}-\mathbf{A}\tilde{\boldsymbol{\Delta}}$ singular, where the subscript $\boldsymbol{\Delta}$ is used to explicitly denote the structure. Braatz *et al.* [14, 15] and Fu [39] have shown that in the general case, the determination of the exact value or an approximate bound on μ is not computationally tractable; however this is not a serious limitation as a tight upper bound on μ for complex structured perturbations can be readily computed [35]. For notational simplicity, consider the perturbation set consisting of square matrices and let $\mathcal{D}$ be set of matrices that commute with all elements of $\mathbf{\Delta}$ or $\mathbf{D}\tilde{\mathbf{\Delta}} = \tilde{\mathbf{\Delta}}\mathbf{D}$ for all $\tilde{\mathbf{\Delta}} \in \mathbf{\Delta}, \mathbf{D} \in \mathcal{D}$. Then,

$$\mu_{\mathbf{\Delta}}(\mathbf{A}) \leq \inf_{\mathbf{D}\in\mathcal{D}} \bar{\sigma}(\mathbf{DAD}^{-1}) \tag{3.1}$$

In this thesis, we denote the upper bound given by (3.1) as $\bar{\mu}_{\mathbf{\Delta}}(.)$. The upper bound given by (3.1) is tight for complex perturbations and the equality holds, if the number of blocks in $\mathbf{\Delta}$ is less than 4. When $\mathbf{\Delta}$ has 4 or more blocks with no block being a repeated scalar, the ratio of $\mu_{\mathbf{\Delta}}(.)$ and $\bar{\mu}_{\mathbf{\Delta}}(.)$ for the worst known example is 0.85 and is close to 1 for most cases [117]. A collection of many useful properties of the structured singular value can be found in [102, 117]. One particularly useful property of the structured singular value, which is used later in this chapter is:

$$\mu_{\mathbf{\Delta}}\left(\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{bmatrix}\right) = \sqrt{\bar{\sigma}(\mathbf{A})\bar{\sigma}(\mathbf{B})} \tag{3.2}$$

where $\mathbf{\Delta} = \text{diag}(\mathbf{\Delta}_1, \mathbf{\Delta}_2)$ and $\mathbf{\Delta}_1, \mathbf{\Delta}_2$ are full complex matrices [100].

3.2.2 Optimal Hankel Norm Approximation

For practical controller design and system identification, use of low order controllers or models is preferred because of online implementation issues. Imposing an order constraint directly on the controller design or identification algorithms usually makes the problem non-convex and difficult to solve. To avoid this difficulty, we can first solve the unrestricted problem (wrt the controller or model order) and then reduce the order of the solution using efficient techniques. In this subsection, we discuss such an order reduction technique, *i.e.*, Hankel norm approximation approach.

Let $\mathbf{G}(s)$ be a stable, square and rational transfer matrix having order n. The objective is to find a reduced order stable transfer matrix $\hat{\mathbf{G}}^k(s)$ having order k such that

$$\hat{\mathbf{G}}^k(s) = \arg \min_{\hat{\mathbf{G}}^k(s)} \|\mathbf{G}(s) - \hat{\mathbf{G}}^k(s)\|_H$$

where $\|.\|_H$ denotes the Hankel norm. This problem has been studied by many researchers and a complete solution is given by Glover [42], who showed that

$$\begin{aligned} \min_{\hat{\mathbf{G}}^k(s)\in\mathcal{RH}_\infty} \|\mathbf{G}(s) - \hat{\mathbf{G}}^k(s)\|_H &= \min_{\hat{\mathbf{G}}^k(s),\mathbf{F}^*(-s)\in\mathcal{RH}_\infty} \|\mathbf{G}(s) - \hat{\mathbf{G}}^k(s) - \mathbf{F}(s)\|_\infty \quad (3.3) \\ &= \sigma_{H,k+1}(\mathbf{G}(s)) \quad (3.4) \end{aligned}$$

where $\sigma_{H,k+1}(.)$ denotes the $(k+1)^{th}$ Hankel singular value. A complete characterization of all solutions that achieve the lowest approximation error (3.4) is available in the original paper by Glover [42] and many standard optimal control textbooks. Note that the requirement that $\mathbf{G}(s)$ be square is not restrictive and is easily satisfied by padding extra zero columns or rows on the non-square $\mathbf{G}(s)$. When the order of the approximation k is zero, $\hat{\mathbf{G}}^k(s)$ is a constant matrix and thus (3.3) is equivalent to finding an anti-stable approximation of a stable system. In this case, the Hankel norm approximation problem is alternatively known as the Nehari extension problem [37].

For the block diagonal approximation problem discussed later in the chapter, we require that $\mathbf{G}(s)$ with n_p unstable poles be approximated by $\hat{\mathbf{G}}^k(s)$ with k unstable poles such that $\|\mathbf{G}(s) - \hat{\mathbf{G}}^k(s)\|_\infty$ is minimized. Next, we show that this problem can also be solved as a Hankel norm approximation problem.

Let $\mathbf{G}(s) = \mathbf{G}_1(s) + \mathbf{G}_2(s)$ such that $\mathbf{G}_1^*(-s), \mathbf{G}_2(s) \in \mathcal{RH}_\infty$. Without loss of generality, we can parameterize $\hat{\mathbf{G}}_k(s)$ as $\hat{\mathbf{G}}^k(s) = \hat{\mathbf{G}}_1^k(s) + \mathbf{G}_2(s)$, which provides

$$\|\mathbf{G}(s) - \hat{\mathbf{G}}^k(s)\|_\infty = \|\mathbf{G}_1(s) - \hat{\mathbf{G}}_1^k(s)\|_\infty = \|\mathbf{G}_1^*(-s) - (\hat{\mathbf{G}}_1^k(s))^*\|_\infty$$

The optimal value for $(\hat{\mathbf{G}}_1^k(s))^* \in \mathcal{RH}_\infty$ is found by solving (*cf.* (3.3)),

$$\min_{(\hat{\mathbf{G}}_1^k(s))^*, \mathbf{F}^*(-s) \in \mathcal{RH}_\infty} \|\mathbf{G}_1^*(-s) - (\hat{\mathbf{G}}_1^k(s))^* - \mathbf{F}(s)\|_\infty$$

Then, the optimal value of $\hat{\mathbf{G}}^k(s)$ is given as $\hat{\mathbf{G}}^k(s) = \hat{\mathbf{G}}_1^k(s) + \mathbf{F}^*(-s) + \mathbf{G}_2(s)$. Since $\mathbf{F}^*(-s)$ and $\mathbf{G}_2(s)$ are stable, $\hat{\mathbf{G}}^k(s)$ is the $\mathcal{L}_\infty$ optimal reduced order approximation of $\mathbf{G}(s)$ with k unstable poles.

3.2.3 Linear Matrix Inequalities

Many control theoretic problems require solving an optimization problem that does not admit an analytic solution. Solving these problems numerically strongly depends on whether the optimization problem is convex and if not, how closely it can be approximated by an equivalent convex problem. Linear matrix inequalities (LMIs) represent a class of such convex constraints and are represented as follows,

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + \sum_{i=1}^{n} \mathbf{x}_i \mathbf{F}_i \succ 0 \tag{3.5}$$

where $\mathbf{x} \in \mathbb{R}^n$ represents the decision variable and $\mathbf{F}_i \in \mathbb{R}^{n \times n}$ are symmetric matrices. In (3.5), $\succ$ is the partial ordering symbol and $\mathbf{F}(\mathbf{x}) \succ 0$ implies that $\mathbf{F}(\mathbf{x})$ is positive definite. The past decade has seen a rapid growth in the use of LMIs for solving control problems because many non-linear optimization problems can be represented as LMIs that are affine in the decision variables.

In this chapter, we use LMIs as the primary computational tool for solving relevant optimization problems. To fully appreciate the importance of LMIs, consider the problem of calculating $\bar{\mu}_{\Delta}(.)$. Due to the presence of the inverse term in (3.1), the optimization problem is difficult to solve in its present form; however, it can be transformed into an equivalent convex optimization problem. There exists $\mathbf{D} \in \mathcal{D}$ such that $\bar{\mu}_{\Delta}(\mathbf{A}) < \gamma$ iff

$$\begin{aligned} (\mathbf{DAD}^{-1})^*(\mathbf{DAD}^{-1}) &\prec \gamma^2\mathbf{I} \quad \text{for some } \mathbf{D} \in \mathcal{D} \\ \Leftrightarrow \mathbf{A}^*\mathbf{D}^*\mathbf{DA} &\prec \gamma^2\mathbf{D}^*\mathbf{D} \quad \text{for some } \mathbf{D} \in \mathcal{D} \\ \Leftrightarrow \mathbf{A}^*\mathbf{PA} &\prec \gamma^2\mathbf{P} \quad \text{for some } \mathbf{P} = \mathbf{D}^*\mathbf{D}, \mathbf{P} \succ 0, \mathbf{P} \in \mathcal{D} \end{aligned} \tag{3.6}$$

For a given γ, (3.6) is affine in the decision variable $\mathbf{P}$. Thus the minimal value of γ can be found using a bisection search method and $\bar{\mu}_{\Delta}(\mathbf{A}) = \inf \gamma$ such that (3.6) holds. The class of problems having a form similar to (3.6) are known as generalized eigenvalue problems. A collection of many other control problems that can be reduced to the LMI form is available in [10].

Naturally, not every optimization problem can be reduced to LMIs. A more general class of matrix inequality problems is that which involves the product of two decision variables. These inequalities are known as bilinear matrix inequalities (BMIs) and have the following general form,

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{F}_0 + \sum_{i=1}^{n} \mathbf{x}_i\mathbf{F}_i + \sum_{j=1}^{n} \mathbf{y}_j\mathbf{G}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{x}_i\mathbf{y}_j\mathbf{H}_{ij} \succ 0 \tag{3.7}$$

where $\mathbf{F}_i, \mathbf{G}_j, \mathbf{H}_{ij} \succ 0$ for all i, j and $\mathbf{y} \in \mathbb{R}^n$. These BMIs are much more difficult to solve than LMIs and are known to be computationally intractable [108]. When one of the decision variables in (3.7) is fixed, the relation becomes an LMI. Then, the BMI (3.7) can be sub-optimally solved by iteratively by fixing one of the $\mathbf{x}$ and $\mathbf{y}$ at a time. This simplistic often provides satisfactory solution to the BMI. A survey of other techniques for solving BMIs can be found in [108] and its references.

3.3 μ-Interaction Measure

In this section, we briefly review the available results on μ-IM [49], point to its limitation and suggest a modification to overcome the same. Throughout this chapter, we assume that the system does not contain any decentralized fixed modes [112]. The absence of decentralized fixed modes is both necessary and sufficient for existence of a decentralized stabilizing controller but only necessary, when individual loops of the decentralized controller are designed independently of each other.

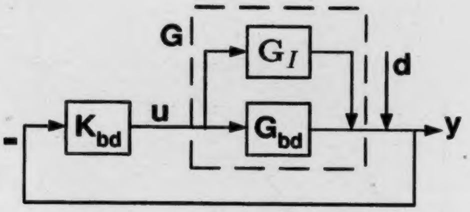


Figure 3.1: Closed loop system

With reference to Figure 3.1, let the system $\mathbf{G}(s)$ be partitioned as $\mathbf{G}(s) = \mathbf{G}_{bd}(s) + \mathbf{G}_I(s)$ such that

- $\mathbf{G}_{bd}(s)$ contains the block-diagonal elements of $\mathbf{G}(s)$ and
- $\mathbf{G}_{bd}(s)$ and $\mathbf{G}(s)$ have the same number of RHP poles.

Define the transfer matrices $\mathbf{E}(s)$ and $\mathbf{T}_{bd}(s)$ as,

$$\mathbf{T}_{bd}(s) = \mathbf{G}_{bd}\mathbf{K}_{bd}(s)\,(\mathbf{I} + \mathbf{G}_{bd}\mathbf{K}_{bd}(s))^{-1} \tag{3.8}$$

$$\mathbf{E}(s) = (\mathbf{G}(s) - \mathbf{G}_{bd}(s))\,\mathbf{G}_{bd}^{-1}(s) \tag{3.9}$$

where $\mathbf{K}_{bd}(s)$ is the block diagonal controller. $\mathbf{T}_{bd}(s)$ can be interpreted as the complementary sensitivity function if $\mathbf{G}_I(s)$ were zero, and $\mathbf{E}(s)$ as the multiplicative uncertainty in $\mathbf{G}_{bd}(s)$. Let $\mathbf{K}_{bd}(s)$ be designed such that $\mathbf{T}_{bd}(s)$ is stable. The central question remains: Does $\mathbf{K}_{bd}(s)$ also stabilize $\mathbf{G}(s)$? This issue has been addressed by Grosdidier and Morari [49], who proposed the use of μ-IM for this purpose.

Lemma 3.1 Assume that $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ have same number of RHP poles and $\mathbf{T}_{bd}(s)$ is stable. Then $\mathbf{T}(s) = \mathbf{G}\mathbf{K}_{bd}(s)\,(\mathbf{I} + \mathbf{G}\mathbf{K}_{bd}(s))^{-1}$ is stable iff the following conditions hold [49]

$$\det(\mathbf{I} + \mathbf{E}\mathbf{T}_{bd}(s)) \neq 0 \tag{3.10}$$

$$N(0, \det(\mathbf{I} + \mathbf{E}\mathbf{T}_{bd}(s))) = 0 \tag{3.11}$$

where $N(\alpha, .)$ denotes the winding number [110] or the number of clockwise encirclements of the point $(\alpha, 0)$ by the image of Nyquist D contour under (.).

Proof: The return difference transfer function for $\mathbf{T}(s)$ can be written as,

$$\begin{aligned}(\mathbf{I} + \mathbf{G}\mathbf{K}_{bd}(s)) &= (\mathbf{I} + \mathbf{G}_{bd}\mathbf{K}_{bd}(s) + \mathbf{G}_I\mathbf{K}_{bd}(s)) \\ &= \left(\mathbf{I} + \mathbf{G}_I\mathbf{K}_{bd}(s)(\mathbf{I} + \mathbf{G}_{bd}\mathbf{K}_{bd}(s))^{-1}\right)(\mathbf{I} + \mathbf{G}_{bd}\mathbf{K}_{bd}(s)) \\ &= (\mathbf{I} + \mathbf{E}\mathbf{T}_{bd}(s))\,(\mathbf{I} + \mathbf{G}_{bd}\mathbf{K}_{bd}(s))\end{aligned} \tag{3.12}$$

Since for rational systems $g_1(s), g_2(s)$, $N(\alpha, g_1g_2(s)) = N(\alpha, g_1(s)) + N(\alpha, g_2(s))$ (see *e.g.* [110]), using the alternate expression for the return difference transfer function (3.12),

$$N(0, \det(\mathbf{I} + \mathbf{G}\mathbf{K}_{bd}(s))) = N\left(0, \det(\mathbf{I} + \mathbf{E}\mathbf{T}_{bd}(s))\right) + N(0, \det(\mathbf{I} + \mathbf{G}_{bd}\mathbf{K}_{bd}(s))) \tag{3.13}$$

Since $\mathbf{G}(s)$ has n_p unstable poles, it follows from the multivariate Nyquist stability criterion [102] that $\mathbf{T}(s)$ is stable iff

$$\begin{aligned} \det(\mathbf{I} + \mathbf{G}(s)\mathbf{K}_{bd}(s)) &\neq 0 \\ N(0, \det(\mathbf{I} + \mathbf{G}(s)\mathbf{K}_{bd}(s))) &= -n_p \end{aligned}$$

Further, since $\mathbf{T}_{bd}(s)$ is stable by assumption,

$$\begin{aligned} \det(\mathbf{I} + \mathbf{G}_{bd}(s)\mathbf{K}_{bd}(s)) &\neq 0 \\ N(0, \det(\mathbf{I} + \mathbf{G}_{bd}(s)\mathbf{K}_{bd}(s))) &= -n_p \end{aligned}$$

The necessity and sufficiency of (3.10)-(3.11) follows using above expressions and (3.13). ■

Lemma 3.1 was originally proven by Grosdidier and Morari [49], except the requirement that (3.10) must hold. This is a minor technical requirement to ensure that the image of $\det(\mathbf{I} + \mathbf{T}_{bd}\mathbf{E}(s))$ does not pass through the origin of the complex plane. Lemma 3.1 forms the basis for a more important result, as presented next.

Theorem 3.1 Let $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ have same number of unstable poles. If $\mathbf{K}_{bd}(s)$ stabilizes $\mathbf{G}_{bd}(s)$, then $\mathbf{K}_{bd}(s)$ also stabilizes $\mathbf{G}(s)$, if

$$\bar{\sigma}\left(\mathbf{T}_{bd}(j\omega)\right) < \mu_{\mathbf{\Delta}}^{-1}\left(\mathbf{E}(j\omega)\right) \quad \forall \omega \in \mathbb{R} \tag{3.14}$$

where $\mathbf{\Delta}$ has the same block structure as $\mathbf{G}_{bd}(s)$ and $\mathbf{T}_{bd}(s)$, $\mathbf{E}(s)$ are defined by (3.8) and (3.9) respectively.

Proof: The sufficiency of (3.14) for closed loop stability is proven by contradiction. Let $N(0, \det(\mathbf{I} + \mathbf{E}\mathbf{T}_{bd}(s))) > 0$ and let the image of $\det(\mathbf{I} + \mathbf{E}\mathbf{T}_{bd}(s))$ intersect the negative real axis of complex plane at the frequency ω_o. Then there exists a β, $|\beta| < 1$ such that

$$\det(\mathbf{I} + \beta\mathbf{E}\mathbf{T}_{bd}(j\omega_o)) = 0$$

Similarly, let there exists a frequency ω_1 such that

$$\det(\mathbf{I} + \mathbf{E}\mathbf{T}_{bd}(j\omega_1)) = 0$$

Combining these two conditions, we notice that $\mathbf{T}(s)$ is unstable iff there exists a β, $|\beta| \leq 1$ such that $\det(\mathbf{I} + \beta\mathbf{E}\mathbf{T}_{bd}(j\omega)) = 0$ for some $\omega \in \mathbb{R}$. It follows from the definition of the structured singular value that the norm of smallest perturbation that destabilizes $\mathbf{E}(j\omega)$ is given as $\bar{\sigma}^{-1}(\beta\mathbf{T}_{bd}(j\omega))$. When (3.14) holds, a β, $|\beta| < 1$ such that $\det(\mathbf{I} + \beta\mathbf{E}\mathbf{T}_{bd}(j\omega)) = 0$ for some $\omega \in \mathbb{R}$ does not exist and the closed loop system is stable. ■

Theorem 3.1 was proven by Grosdider and Morari [49] under the requirement that the unstable poles of $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ be identical. It is clear from Lemma 3.1 and the proof of Theorem 3.1 that the number of unstable poles of $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ being equal suffices. In either case, design of $\mathbf{K}_{bd}(s)$ solely based on $\mathbf{G}_{bd}(s)$ is equivalent to designing individual loops independently. The equation (3.14) is known as the μ-IM. This powerful result allows the designer to impose restrictions on individual controllers, but still be designed solely based on $\mathbf{G}_{bd}(s)$ such that closed loop stability is ensured.

As pointed by Grosdidier and Morari [49] that in practice, $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ as defined above have same number of RHP poles only for open loop stable systems limiting the applicability of μ-IM. It is noted that this limitation only arises as $\mathbf{G}_{bd}(s)$ is chosen as the block diagonal elements of $\mathbf{G}(s)$ and is easily overcome by relaxing this requirement. The decentralized controller can be designed based on $\mathbf{G}_{bd}(s)$ that is different from the block diagonal elements but has the same number of RHP poles as $\mathbf{G}(s)$. This point is further illustrated using the following simple system:

$$\mathbf{G}(s) = \frac{1}{(s-1)(s-2)}\begin{bmatrix} (s+0.5) & 0.5 \\ (9s-3) & (s+1) \end{bmatrix} \tag{3.15}$$

Since all the minors of order 1 have $(s-1)(s-2)$ as denominator and

$$\det(\mathbf{G}(s)) = \frac{(s+0.5)(s+1) - 0.5(9s-3)}{(s-1)^2(s-2)^2} = \frac{s^2-3s+2}{(s-1)^2(s-2)^2} = \frac{1}{(s-1)(s-2)}$$

the system (3.15) has two unstable poles at 1 and 2 [81]. Let $\mathbf{G}_{bd}(s)$ be chosen as the diagonal elements of $\mathbf{G}(s)$. In this case,

$$\det(\mathbf{G}_{bd}(s)) = \frac{(s+0.5)(s+1)}{(s-1)^2(s-2)^2}$$

Due to absence of pole-zero cancellation, $\mathbf{G}_{bd}(s)$ has poles at the same locations as $\mathbf{G}(s)$, but repeated twice and the assumption of $\mu-$IM are violated. Consider that $\mathbf{G}_{bd}(s)$ is chosen as,

$$\mathbf{G}_{bd}(s) = \begin{bmatrix} \frac{1}{(s-\alpha_1)}f_1(s) & 0 \\ 0 & \frac{1}{(s-\alpha_2)}f_2(s) \end{bmatrix}$$

where $\alpha_1, \alpha_2 > 0$ and $f_1(s), f_2(s)$ are arbitrary stable transfer matrices. With this choice, the assumption that $\mathbf{G}_{bd}(s)$ and $\mathbf{G}(s)$ have same unstable poles is easily satisfied. Note that for an arbitrary choice of $\alpha_1, \alpha_2 > 0$, the diagonal blocks of $\mathbf{G}_I(s)$ are not necessarily zero. A similar approach can be used for partitioning any arbitrary system.

Remark 3.1 The approach for choosing $\mathbf{G}_{bd}$, as illustrated above, still holds when some of the RHP poles of the system do not appear in any of its block diagonal elements. It is pointed out however that in this case, it may be very difficult to design a block diagonal controller $\mathbf{K}_{bd}$ to satisfy the $\mu-$IM condition, as the corresponding diagonal blocks will have large element-wise uncertainties associated with them (up to 100%, if the diagonal block is $\mathbf{0}$).

Though the generalization used in choosing $\mathbf{G}_{bd}(s)$ extends the practical applicability of $\mu-$IM to unstable systems, the generalization introduces an additional degree of freedom. Clearly, whether the $\mu-$IM condition (3.14) is satisfied depends on the choice of $\mathbf{G}_{bd}(s)$, which is dealt with in subsequent sections.

3.4 Alternate Representation of μ-IM

For a given $\mathbf{G}_{bd}(s)$, a loop shaping approach can be used to find $\mathbf{K}_{bd}(s)$ for closed loop stability. In the present case, $\mathbf{G}_{bd}(s)$ can also be treated as a free parameter with the requirement of having the same number of unstable poles as $\mathbf{G}(s)$.

The task of jointly finding the pair $(\mathbf{G}_{bd}(s), \mathbf{K}_{bd}(s))$ such that the closed loop system is stable, is very difficult. We note in (3.14), both $\bar{\sigma}(\mathbf{T}_{bd}(j\omega))$ and $\mu_{\boldsymbol{\Delta}}(\mathbf{E}(j\omega))$ depend on $\mathbf{G}_{bd}(j\omega)$, but $\mathbf{E}(j\omega)$ is independent of the controller. Then, a convenient (and not optimal) approach is to find $\mathbf{G}_{bd}(s)$ such that $\mu_{\boldsymbol{\Delta}}(\mathbf{E}(j\omega))$ is minimized and then design the decentralized controlled based on it to satisfy the μ-IM condition; however, $\mathbf{E}(s)$ in not an affine function of $\mathbf{G}_{bd}(s)$. We next show that this difficulty can be overcome by representing μ-IM alternately in terms of transfer matrix between the disturbances and the inputs.

Proposition 3.1 Let $\mathbf{G}(s)$ be partitioned as $\mathbf{G}(s) = \mathbf{G}_{bd}(s) + \mathbf{G}_I(s)$ such that $\mathbf{G}_{bd}(s)$ and $\mathbf{G}(s)$ have the same number of RHP poles. Define $\mathbf{S}_{bd}(s) = (\mathbf{I} + \mathbf{G}_{bd}\mathbf{K}_{bd}(s))^{-1}$. Then $\mathbf{K}_{bd}(s)$ stabilizing $\mathbf{G}_{bd}(s)$ also stabilizes $\mathbf{G}(s)$ if

$$\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) < \mu_{\boldsymbol{\Delta}}^{-1}(\mathbf{G}_I(j\omega)) \qquad \forall \omega \in \mathbb{R} \tag{3.16}$$

where $\boldsymbol{\Delta}$ has the same structure as $\mathbf{G}_{bd}(s)$.

Proof: Note that

$$\det(\mathbf{I} + \mathbf{E}\mathbf{T}_{bd}(s)) = \det(\mathbf{I} + \mathbf{G}_I(s)\mathbf{K}_{bd}\mathbf{S}_{bd}(s))$$

Now the sufficiency of (3.16) is shown by using Lemma 3.1 and following the proof of Theorem 3.1. ■

Since the RHS of (3.16) is affine in $\mathbf{G}_{bd}(s)$, it can be sub-optimally selected by minimizing $\mu_\Delta(\mathbf{G}_I(j\omega))$. This approach is suboptimal as the LHS of (3.16) also depends on $\mathbf{G}_{bd}(s)$. For a particular choice of $\mathbf{G}_{bd}(s)$ that optimally minimizes $\mu_\Delta(\mathbf{G}_I(j\omega))$, there may not exist any controller satisfying (3.16). This issue is further discussed later in this chapter.

Remark 3.2 Since both of (3.14) and (3.16) are sufficient but not necessary conditions for closed loop stability, some stabilizing controller may fail to satisfy (3.14) and (3.16) simultaneously. Note that $\det(\mathbf{I}+\mathbf{E}(s)\mathbf{T}_{bd}(s)) = \det(\mathbf{I}+\mathbf{E}(s)\mathbf{W}^{-1}(s)\mathbf{W}(s)\mathbf{T}_{bd}(s))$. Then, a sufficient condition for closed loop stability is that $\bar{\sigma}(\mathbf{W}\mathbf{T}_{bd}(j\omega)) \le \mu_{\mathbf{\Delta}}^{-1}(\mathbf{E}\mathbf{W}^{-1}(j\omega))$ for all $\omega \in \mathbb{R}$ [49]. From the discussion in § 2.3, it follows that we can select $\mathbf{W}(s)$ to have the unstable poles and pole directions as $\mathbf{G}_{bd}(s)$. Clearly, the allowable class of $\mathbf{W}(s)$ includes $\mathbf{G}_{bd}(s)$ itself. Then (3.16) can be seen as a special case of the generalized inequality (3.14). Similarly, (3.14) can be shown to be a special case of the generalized inequality (3.16) using similar arguments.

The modified $\mu-$IM condition (3.16) is derived by treating $\mathbf{K}_{bd}\mathbf{S}_{bd}(s)$ as uncertainty in $\mathbf{G}_I(s)$. A slightly weaker version of (3.16) can be derived by instead considering the robust stabilization of $\mathbf{G}_{bd}(s)$ and using the results of Glover [43], which are useful for analyzing robust stability in presence of unstructured perturbations. In the present context, such an exercise is redundant but can provide insight into the conservatism or more precisely lack of conservatism of $\mu-$IM.

Since $\mathbf{G}_{bd}(s)$ and $\mathbf{G}(s)$ have the same RHP poles, $\mathbf{K}_{bd}(s)$ stabilizing $\mathbf{G}_{bd}(s)$ also stabilizes $\mathbf{G}(s)$ if [43]

$$\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) < \bar{\sigma}^{-1}(\mathbf{G}_I(j\omega)) \quad \forall\omega \tag{3.17}$$

Since stability is scaling invariant, the closed loop system is stable if

$$\bar{\sigma}(\mathbf{D}_L(\omega)\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)\mathbf{D}_R^{-1}(\omega)) < \bar{\sigma}(\mathbf{D}_L(\omega)\mathbf{G}_I(j\omega)\mathbf{D}_R^{-1}(\omega)) \quad \forall\omega \tag{3.18}$$

where $\mathbf{D}_L(\omega)$ and $\mathbf{D}_R(\omega)$ are frequency dependent scaling matrices. Let $\mathbf{D}_L(\omega), \mathbf{D}_R(\omega)$ be restricted to the set

$$\begin{aligned}\mathcal{D}_L &= \{\mathrm{diag}(d_i \cdot \mathbf{I}_{m_i}), d_i \in \mathbb{R}\} \\ \mathcal{D}_R &= \{\mathrm{diag}(d_j \cdot \mathbf{I}_{m_j}), d_j \in \mathbb{R}\}\end{aligned} \tag{3.19}$$

where the dimensions of individual blocks of $\mathbf{G}_{bd}(s)$ is $m_i \times m_j$. Since $\mathbf{D}_L(\omega)\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)\mathbf{D}_R^{-1}(\omega) = \mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)$, the conservatism of (3.18) is reduced by choosing $\mathbf{D}_L(\omega), \mathbf{D}_R(\omega)$ to maximize the RHS of (3.18) at every frequency. Then the sufficient condition for the stability of closed loop system is

$$\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) < \sup_{\substack{\mathbf{D}_L(\omega)\in\mathcal{D}_L \\ \mathbf{D}_R(\omega)\in\mathcal{D}_R}} \bar{\sigma}(\mathbf{D}_L(\omega)\mathbf{G}_I(j\omega)\mathbf{D}_R^{-1}(\omega)) < \bar{\mu}_{\Delta}^{-1}(\mathbf{G}_I(j\omega)) \quad \forall\omega \tag{3.20}$$

Theoretically, (3.20) is slightly more conservative than (3.16). However, from computational point of view, they are equivalent as, in practice, only the upper and lower bounds on μ are computable.

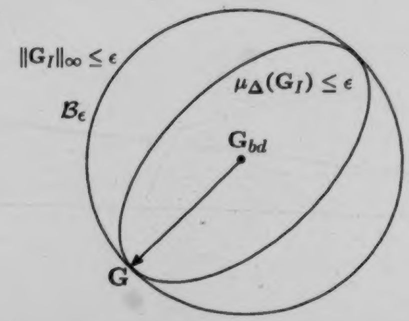

$\|\mathbf{G}_I\|_\infty \leq \epsilon$
$\mathcal{B}_\epsilon$
$\mu_\Delta(\mathbf{G}_I) \leq \epsilon$
$\mathbf{G}_{bd}$
$\mathbf{G}$

Figure 3.2: Physical interpretation of reducing conservatism through μ

Remark 3.3 Most of the available interaction measures other than μ–IM, *e.g.* [78, 88], provide a condition that is equivalent to (3.17). Since $\bar{\sigma}(\mathbf{G}_I(j\omega)) \geq \bar{\mu}_\Delta(\mathbf{G}_I(j\omega)) \geq \mu_\Delta(\mathbf{G}_I(j\omega))$ for all ω, (3.17) is more conservative than (3.20) and (3.16). We can also interpret this result on physical grounds as follows: An uncertainty set $\bar{\sigma}(\mathbf{G}_I(j\omega)) \leq \epsilon(\omega)$ defines an open ball in the complex plane, denoted as $\mathcal{B}_\epsilon$. In this case, the controller needs to stabilize all systems that lie within $\mathcal{B}_\epsilon$ to guarantee that $\mathbf{G}(s)$ is also stabilized. When $\mathcal{B}_\epsilon$ is optimally scaled at every frequency, the dimensions of this perturbation set are shrunk in all directions, except the direction connecting the nominal model $\mathbf{G}_{bd}(j\omega)$

and $\mathbf{G}(j\omega)$. The optimal scaling reduces the number of additional systems that need to be stabilized to guarantee that $\mathbf{G}(s)$ is also stabilized and hence the reduction in conservatism (see Figure 3.2).

Remark 3.4 Compared to the necessary and sufficient conditions provided by Lemma 3.1, the conditions provided by Theorem 3.1 and Proposition 3.1 are sufficient only. To illustrate this point, consider a controller $\mathbf{K}_{bd}(s)$ that violates (3.14) or (3.16), but the closed loop system is stable. Then, there exists some other controller $\bar{\mathbf{K}}_{bd}(s)$ such that $\bar{\sigma}(\bar{\mathbf{K}}_{bd}(j\omega)(\mathbf{I} + \mathbf{G}_{bd}\bar{\mathbf{K}}_{bd}(j\omega))^{-1}) = \bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega))$ for some $\omega \in \mathbb{R}$ and $\bar{\mathbf{K}}_{bd}(s)(\mathbf{I} + \mathbf{G}_{bd}\bar{\mathbf{K}}_{bd}(s))^{-1}$ is unstable. We can also interpret (3.16) as a sufficient condition for robust stabilization of $\mathbf{G}_{bd}(s)$. Similar as before, consider that a controller violates (3.16), but the closed loop system is stable. Then, there exists $\bar{\mathbf{G}}_I(s)$ such that $\bar{\mu}_\Delta(\bar{\mathbf{G}}_I(j\omega)) = \bar{\mu}_\Delta(\mathbf{G}_I(j\omega))$ for some $\omega \in \mathbb{R}$ and the closed loop system is unstable when $\mathbf{G}_I(s)$ is replaced by $\bar{\mathbf{G}}_I(s)$. Thus, the conservativeness of μ-IM arises as the apparent uncertainty set is much larger than the true uncertainty set, which consists of a single element, *i.e.* $\mathbf{G}_I(s)$. The strength of μ-IM is that when (3.14) or (3.16) hold, any decentralized controller that stabilizes $\mathbf{G}_{bd}(s)$ also stabilizes $\mathbf{G}(s)$.

Proposition 3.1 provides a sufficient condition to assess whether $\mathbf{K}_{bd}$ designed for $\mathbf{G}_{bd}$, can stabilize the closed loop system; however, it provides no information regarding the closed loop performance. Grosdidier and Morari [49] pointed out, satisfying μ-IM condition guarantees closed loop stability, but the performance can be arbitrarily poor. In the next proposition, we show that when the μ-IM condition (3.16) is satisfied, an upper bound on closed loop input performance is always minimized.

Proposition 3.2 Assume that $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ have the same number of RHP poles and (3.20) holds. Then,

$$\mu_{\mathbf{\Delta}}(\mathbf{K}_{bd}\mathbf{S}(j\omega)) \le \frac{1}{\bar{\sigma}^{-1}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) - \bar{\mu}_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))} \quad \forall\omega \tag{3.21}$$

where $\mathbf{\Delta}$ has the same structure as $\mathbf{G}_{bd}$.

Proof: Using $\mathbf{G}(s) = \mathbf{G}_{bd}(s) + \mathbf{G}_I(s)$,

$$\begin{aligned}(\mathbf{I} + \mathbf{G}\mathbf{K}_{bd}(s))\mathbf{K}_{bd}^{-1}(s) &= \mathbf{K}_{bd}^{-1}(s) + \mathbf{G}_{bd}(s) + \mathbf{G}_I(s)\\ &= (\mathbf{I} + \mathbf{G}_{bd}\mathbf{K}_{bd}(s))\mathbf{K}_{bd}^{-1}(s) + \mathbf{G}_I(s)\end{aligned} \tag{3.22}$$

Let $\mathbf{D}_L(\omega) \in \mathcal{D}_L$ and $\mathbf{D}_R(\omega) \in \mathcal{D}_R$, where $\mathcal{D}_L$ and $\mathcal{D}_R$ are defined by (3.19). Then, using (3.22) and singular value inequalities [63, 102],

$$\underline{\sigma}\left(\mathbf{D}_L(\omega)\mathbf{S}^{-1}\mathbf{K}_{bd}^{-1}(j\omega)\mathbf{D}_R^{-1}(\omega)\right) \geq \underline{\sigma}\left(\mathbf{D}_L(\omega)\mathbf{S}_{bd}^{-1}\mathbf{K}_{bd}^{-1}(j\omega)\mathbf{D}_R^{-1}(\omega)\right) - \bar{\sigma}\left(\mathbf{D}_L(\omega)\mathbf{G}_I(j\omega)\mathbf{D}_R^{-1}(\omega)\right)$$

Consider that $\mathbf{D}_L(\omega), \mathbf{D}_R(\omega)$ are chosen to maximize $\bar{\sigma}\left(\mathbf{D}_L(\omega)\mathbf{G}_I(j\omega)\mathbf{D}_R^{-1}(\omega)\right)$. Since $\mathbf{D}_L(\omega)\mathbf{S}_{bd}^{-1}\mathbf{K}_{bd}^{-1}(j\omega)\mathbf{D}_R^{-1}(\omega) = \mathbf{S}_{bd}^{-1}\mathbf{K}_{bd}^{-1}(j\omega)$,

$$\underline{\sigma}\left(\mathbf{D}_L(\omega)\mathbf{S}^{-1}\mathbf{K}_{bd}^{-1}(j\omega)\mathbf{D}_R^{-1}(\omega)\right) \geq \underline{\sigma}\left(\mathbf{S}_{bd}^{-1}\mathbf{K}_{bd}^{-1}(j\omega)\right) - \bar{\mu}_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega)) \tag{3.23}$$

With this choice, $\mu_{\mathbf{\Delta}}^{-1}(\mathbf{K}_{bd}\mathbf{S}(j\omega)) \geq \underline{\sigma}\left(\mathbf{D}_L(\omega)\mathbf{S}^{-1}\mathbf{K}_{bd}^{-1}(j\omega)\mathbf{D}_R^{-1}(\omega)\right)$ and

$$\begin{aligned} \mu_{\mathbf{\Delta}}^{-1}(\mathbf{K}_{bd}\mathbf{S}(j\omega)) &\geq \underline{\sigma}\left(\mathbf{S}_{bd}^{-1}\mathbf{K}_{bd}^{-1}(j\omega)\right) - \bar{\mu}_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega)) \\ \mu_{\mathbf{\Delta}}(\mathbf{K}_{bd}\mathbf{S}(j\omega)) &\leq \frac{1}{\bar{\sigma}^{-1}\left(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)\right) - \bar{\mu}_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))} \end{aligned}$$

■

Generally, the nominal performance of the closed loop system is measured in terms of $\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}(j\omega))$ instead of $\mu_{\mathbf{\Delta}}(\mathbf{K}_{bd}\mathbf{S}(j\omega))$. The corollary below shows that the information regarding $\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}(j\omega))$ can be readily extracted from (3.21).

Corollary 3.1 Let all the conditions of Proposition 3.2 hold and $\mathbf{D}_L(\omega) \in \mathcal{D}_L$, $\mathbf{D}_R(\omega) \in \mathcal{D}_R$ be chosen to maximize $\bar{\sigma}\left(\mathbf{D}_L(\omega)\mathbf{G}_I(j\omega)\mathbf{D}_R^{-1}(\omega)\right)$ Then

$$\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}(j\omega)) \leq \frac{\kappa(\mathbf{D}_L(\omega))}{\bar{\sigma}^{-1}\left(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)\right) - \bar{\mu}_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))} \forall \omega \in \mathbb{R} \tag{3.24}$$

where $\mathbf{\Delta}$ has same structure as $\mathbf{G}_{bd}$ and κ denotes the Euclidean condition number.

Proof: Using (3.23),

$$\bar{\sigma}(\mathbf{D}_L(\omega))\bar{\sigma}(\mathbf{D}_R^{-1}(\omega))\underline{\sigma}\left(\mathbf{S}^{-1}\mathbf{K}_{bd}^{-1}(j\omega)\right) \geq \underline{\sigma}\left(\mathbf{S}_{bd}^{-1}\mathbf{K}_{bd}^{-1}(j\omega)\right) - \bar{\mu}_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega)) \;\; \forall \omega \tag{3.25}$$

Since $\bar{\sigma}(\mathbf{D}_R^{-1}(\omega)) = \bar{\sigma}(\mathbf{D}_L^{-1}(\omega))$ by construction, $\bar{\sigma}(\mathbf{D}_L(\omega))\bar{\sigma}(\mathbf{D}_R^{-1}(\omega)) = \kappa(\mathbf{D}_L(\omega))$. With this observation, (3.24) can be obtained by rearranging (3.25) as the proof of Proposition 3.2. ■

Comparing (3.25) with (3.20), we notice that when the decentralized controller stabilizes the closed loop system, an upper bound on the closed loop input performance is always minimized. The bound on the closed loop performance (3.25) is very loose in general. When the performance requirements are specified in terms of a frequency dependent weight, it can be very difficult to satisfy these requirements by minimizing the upper bound. Nevertheless, maximization of $\bar{\sigma}^{-1}\left(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)\right) - \bar{\mu}_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))$ is beneficial to maximize the robustness of the closed loop system for unmodelled dynamics that can be represented as an additive uncertainty [43].

3.5 Block Diagonal Approximation

In this section, we consider the problem of finding an optimal block diagonal approximation $\mathbf{G}_{bd}(s)$ for the given system $\mathbf{G}(s)$ such that $\mu_\Delta(\mathbf{G}(j\omega) - \mathbf{G}_{bd}(j\omega))$ is minimized. Since only $\bar{\mu}_\Delta(.)$ is computable in practice, the block diagonal $\mathbf{G}_{bd}(s)$ can be chosen by solving,

$$\begin{aligned} &\min_{\mathbf{G}_{bd}(j\omega)} \bar{\sigma}\left(\mathbf{D}_L(\omega)(\mathbf{G}(j\omega) - \mathbf{G}_{bd}(j\omega))\mathbf{D}_R^{-1}(\omega)\right) \\ \text{s.t.} \quad &\mathbf{D}_L(\omega) \in \mathcal{D}_L, \mathbf{D}_R(\omega) \in \mathcal{D}_R \end{aligned} \tag{3.26}$$

where $\mathcal{D}_L$ and $\mathcal{D}_R$ are given by (3.19) and the number of unstable poles of $\mathbf{G}_{bd}(s)$ and $\mathbf{G}(s)$ is same.

Intuitively, a suboptimal solution to the optimization problem (3.26) can be obtained by simply reducing the order of the block diagonal elements of $\mathbf{G}(s)$. In fact, for systems decomposed into 2 blocks, the solution obtained by order reduction of the diagonal elements is optimal. This result is proven next by showing that the diagonal blocks optimally approximate a complex matrix partitioned into 2 blocks, which may also be of independent interest.

Proposition 3.3 Consider a complex matrix $\mathbf{A} \in \mathbb{C}^{p\times q}$ be partitioned as,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

Then, $\mathbf{A}_{bd} = \operatorname{diag}(\mathbf{A}_{11}, \mathbf{A}_{22})$ minimizes $\mu_{\mathbf{\Delta}}(\mathbf{A} - \mathbf{A}_{bd})$, where $\mathbf{A}_{bd}$ and $\mathbf{\Delta}$ have the same structure as $\operatorname{diag}(\mathbf{A}_{11}, \mathbf{A}_{22})$ and

$$\min_{\mathbf{A}_{bd}} \mu_{\mathbf{\Delta}}(\mathbf{A} - \mathbf{A}_{bd}) = \sqrt{\bar{\sigma}(\mathbf{A}_{12})\bar{\sigma}(\mathbf{A}_{21})} \tag{3.27}$$

Proof: Using the identity for structured singular value (3.2), it follows that $\mu_{\mathbf{\Delta}}(\mathbf{A} - \operatorname{diag}(\mathbf{A}_{11}, \mathbf{A}_{22})) = \sqrt{\bar{\sigma}(\mathbf{A}_{12})\bar{\sigma}(\mathbf{A}_{21})}$. Then, it suffices to show that for all $\mathbf{A}_{bd}$, the minimum achievable value of $\mu_{\mathbf{\Delta}}(\mathbf{A} - \mathbf{A}_{bd})$ is given by (3.27).

Let $\mathbf{A}_{bd} = \operatorname{diag}(\mathbf{A}_{11} + \mathbf{B}_1, \mathbf{A}_{22} + \mathbf{B}_2)$. Since $\mathbf{\Delta}$ has two complex blocks,

$$\begin{aligned} \mu_{\mathbf{\Delta}}(\mathbf{A} - \mathbf{A}_{bd}) &= \inf_{\mathbf{D}_L \in \mathcal{D}_L, \mathbf{D}_R \in \mathcal{D}_R} \bar{\sigma}(\mathbf{D}_L(\mathbf{A} - \mathbf{A}_{bd})\mathbf{D}_R^{-1}) \\ &= \inf_{d_1, d_2 \in \mathbb{R}} \bar{\sigma}\left(\begin{bmatrix} \mathbf{B}_1 & \frac{d_1}{d_2}\mathbf{A}_{12} \\ \frac{d_2}{d_1}\mathbf{A}_{21} & \mathbf{B}_2 \end{bmatrix}\right) \end{aligned}$$

Let $\mathbf{U}$ be a unitary matrix that permutes the off-diagonal blocks of $\mathbf{D}_L(\mathbf{A} - \mathbf{A}_{bd})\mathbf{D}_R^{-1}$ to diagonal blocks and *vice versa*. Without loss of generality, we can choose $d_1 = 1$ [117].

Since the largest singular value of a matrix is larger than or equal to largest singular value of the sub-matrices of the matrix [63],

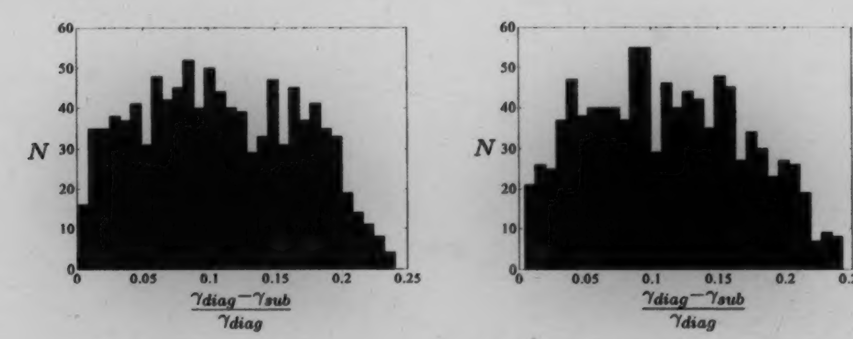
$$\begin{aligned}\bar{\sigma}(\mathbf{D}_L(\mathbf{A}-\mathbf{A}_{bd})\mathbf{D}_R^{-1}) &= \bar{\sigma}(\mathbf{D}_L(\mathbf{A}-\mathbf{A}_{bd})\mathbf{D}_R^{-1}\mathbf{U}) \\ &\geq \max(\bar{\sigma}(d_2^{-1}\mathbf{A}_{12}), \bar{\sigma}(d_2\mathbf{A}_{21})) \;\; \forall d_2 \in \mathbb{R} \\ &\geq \max\left(|d_2^{-1}|\,\bar{\sigma}(\mathbf{A}_{12}), |d_2|\,\bar{\sigma}(\mathbf{A}_{21})\right) \;\; \forall d_1, d_2 \in \mathbb{R} \\ &\geq \sqrt{\bar{\sigma}(\mathbf{A}_{12})\bar{\sigma}(\mathbf{A}_{21})}\end{aligned}$$

The result follows by noting that the RHS of the above expression is independent of the scaling matrices. ■

Note that Proposition 3.3 says nothing about the uniqueness of the optimal solution. For $(\mathbf{A}-\mathbf{A}_{bd})$ partitioned and permuted as done in the proof of Proposition 3.3 [117, p. 218],

$$\mu_{\mathbf{\Delta}}(\mathbf{A}-\mathbf{A}_{bd}) \leq \max(\bar{\sigma}(\mathbf{A}_{12}), \bar{\sigma}(\mathbf{A}_{21})) + \sqrt{\bar{\sigma}(\mathbf{B}_1)\bar{\sigma}(\mathbf{B}_2)}$$

If $\mathbf{B}_1 = \mathbf{0}$ and $\bar{\sigma}(\mathbf{A}_{12}) = \bar{\sigma}(\mathbf{A}_{21})$, the upper bound on $\mu_{\mathbf{\Delta}}(\mathbf{A}-\mathbf{A}_{bd})$ is the same as the lower bound. This shows that there exists an infinite number of $\mathbf{B}_2$ and thus block diagonal matrices which achieve the lower bound.



(a) Unrestricted case

(b) Diagonally dominant matrices

Figure 3.3: Relative difference between approximation errors using diagonal elements (γ_{diag}) and locally optimal solution (γ_{sub}) for 3 × 3 complex matrices

Unfortunately, Proposition 3.3 does not hold for matrices partitioned into more than 2 blocks. For such cases, we may still hope that the diagonal blocks will be nearly optimal for the approximation problem. To verify the extent of sub-optimality of using diagonal blocks as a nearly optimal solution for the approximation problem, 1000 3 × 3 complex matrices are generated randomly. The real and imaginary parts of the individual elements of the matrices lie between ±100. For comparison purposes, the locally optimal solution is calculated using the method discussed in the next subsection. Figure 3.3(a) shows that the relative difference between the approximation errors using diagonal elements and the (locally) optimal solution can be as high as 0.25. When the class of random matrices is limited to the diagonally dominant matrices, surprisingly the same upper bound still holds (see Figure 3.3(b)). Thus, we conclude that the solution obtained by simply reducing the order of the diagonal blocks is restrictive and present an algorithm that provides a locally optimal solution for the optimization problem (3.26).

Algorithm 3.1 For a given system $\mathbf{G}(s)$ with n_p unstable poles, a locally optimal solution to the block diagonal approximation problem is obtained by the following steps:

1. Solve the optimization problem (3.26) at a set of chosen frequencies to yield $\mathbf{G}_{bd,j\omega}$.
2. Solve a parametric optimization problem to find $\tilde{\mathbf{G}}_{bd}(s)$ that has at least n_p unstable poles and minimizes the worst case error between $\tilde{\mathbf{G}}_{bd}(j\omega)$ and $\mathbf{G}_{bd,j\omega}$.
3. If $\tilde{\mathbf{G}}_{bd}(s)$ has more than n_p unstable poles, the order of $\tilde{\mathbf{G}}_{bd}(s)$ is reduced to n_p through optimal Hankel norm approximation to get $\mathbf{G}_{bd}(s)$.

The role of these steps becomes clear by noting,

$$\begin{aligned}\mu_{\mathbf{\Delta}}(\mathbf{G}(j\omega) - \mathbf{G}_{bd}(j\omega)) \le \mu_{\mathbf{\Delta}}(\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega})\\ + \bar{\sigma}(\mathbf{G}_{bd,j\omega} - \tilde{\mathbf{G}}_{bd}(j\omega)) + \bar{\sigma}(\tilde{\mathbf{G}}_{bd}(j\omega) - \mathbf{G}_{bd}(j\omega)) \quad (3.28)\end{aligned}$$

It follows from (3.28) that every step in the proposed method minimizes the contribution of one of terms on RHS of (3.28) to the total approximation error. The order reduction through Hankel norm approximation was discussed in § 3.2.2 and is not repeated. The other steps of the proposed method are discussed next.

3.5.1 Frequency Wise Approximation

The first step of the proposed method for finding the optimal block diagonal approximation consists of minimizing (3.26) at a set of chosen frequencies. The (possibly non-uniformly spaced) set of frequencies can be selected based on $\bar{\sigma}(\mathbf{G}(j\omega))$, *i.e.*, a larger number of frequencies can be chosen around the peaks of $\bar{\sigma}(\mathbf{G}(j\omega))$. In the remaining discussion, the frequency argument of the scaling matrices is dropped for notational convenience. Using similar arguments as used in calculating $\bar{\mu}(.)$ (3.6),

$$\begin{aligned} \bar{\sigma}(\mathbf{D}_L(\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega})\mathbf{D}_R^{-1}) &\leq \gamma & (3.29)\\ \Leftrightarrow \mathbf{D}_R^{-*}(\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega})^*\mathbf{D}_L^*\mathbf{D}_L(\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega})\mathbf{D}_R^{-1} &\preceq \gamma^2\mathbf{I} & (3.30)\\ \Leftrightarrow (\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega})^*\mathbf{P}_L(\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega}) &\preceq \gamma^2\mathbf{P}_R & (3.31) \end{aligned}$$

where $\mathbf{P}_L = \mathbf{D}_L^*\mathbf{D}_L \in \mathcal{D}_L$, $\mathbf{P}_R = \mathbf{D}_R^*\mathbf{D}_R \in \mathcal{D}_R$ and $\mathbf{P}_L, \mathbf{P}_R \succ 0$. Note that unlike (3.6), (3.31) is not affine in the decision variables; however, a locally optimal solution can be found using an iterative approach. Using the Schur complement lemma [10], (3.30) can be equivalently expressed as,

$$\begin{bmatrix} -\gamma\mathbf{I} & \mathbf{D}_R^{-*}(\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega})^*\mathbf{D}_L^* \\ \mathbf{D}_L(\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega})\mathbf{D}_R^{-1} & -\gamma\mathbf{I} \end{bmatrix} \preceq 0 \qquad (3.32)$$

Note that for fixed $\mathbf{D}_L, \mathbf{D}_R$, (3.32) is an LMI in $\mathbf{G}_{bd,j\omega}$. Now, a locally optimal solution for the frequency wise approximation problem can be found by using the following iterative algorithm for the set of chosen frequencies:

Algorithm 3.2 Select a set of frequencies $\{\omega_i\}$, $i = 1 \cdots n_\omega$ and evaluate $\mathbf{G}(j\omega_i)$. Choose convergence tolerance ϵ and initial $\mathbf{D}_L^0 \in \mathcal{D}_L$, $\mathbf{D}_R^0 \in \mathcal{D}_R$ (*e.g.* $\mathbf{D}_L^0 = \mathbf{D}_R^0 = \mathbf{I}$), where $\mathcal{D}_L, \mathcal{D}_R$ are given by (3.19). Set $i = 1$.

1. Solve the convex optimization problem (3.32) for $\mathbf{G}_{bd,j\omega}^i$ by setting $\mathbf{D}_L = \mathbf{D}_L^{i-1}$ and $\mathbf{D}_R = \mathbf{D}_R^{i-1}$. Let the locally optimal approximation error be γ_1^i.

2. Solve (3.31) for $\mathbf{P}_L^i, \mathbf{P}_R^i$ using a bisection search method by fixing $\mathbf{G}_{bd,j\omega}$ as $\mathbf{G}_{bd,j\omega}^i$. Let the locally optimal approximation error be γ_2^i. Set $\mathbf{D}_L^i = (\mathbf{P}_L^i)^{0.5}$, $\mathbf{D}_R^i = (\mathbf{P}_R^i)^{0.5}$ and $i = i + 1$.

3. Repeat steps 1 and 2 until $|\gamma_1^{i-1} - \gamma_2^{i-1}| < \epsilon$.

Unlike a general BMI problem, the sequence of solutions obtained using Algorithm 3.2 is guaranteed to converge. Let $\gamma_1^i, \gamma_2^i, \gamma_1^{i+1}$ be a sequence of the approximation errors. For convergence, we only need to show that $\gamma_1^i \geq \gamma_2^i \geq \gamma_1^{i+1}$. Since the individual optimization problems to be solved in steps 1 and 2 of Algorithm 3.2 are convex, these steps are jointly convex if there always exists $\mathbf{P}_L^i, \mathbf{P}_R^i$ such that $\gamma_1^i = \gamma_2^i$ and $\mathbf{G}_{bd,j\omega}^{i+1}$ such that $\gamma_2^i = \gamma_1^{i+1}$.

Since (3.29) ⇔ (3.32),

$$\gamma_1^i = \bar{\sigma}(\mathbf{D}_L^{i-1}(\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega}^i)(\mathbf{D}_R^{i-1})^{-1})$$

With $\mathbf{G}_{bd,j\omega} = \mathbf{G}_{bd,j\omega}^i$, $\mathbf{P}_L^i = \mathbf{P}_R^i = \mathbf{I}$ is a feasible solution for (3.31). As (3.31) is convex, the solution $\mathbf{P}_L^i = \mathbf{P}_R^i = \mathbf{I}$ can be seen as the worst case solution, which achieves $\gamma_2^i = \gamma_1^i$. Now, note that

$$(\mathbf{D}_L^i(\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega}^{i+1})(\mathbf{D}_R^i)^{-1}) = \mathbf{D}_L^i\mathbf{G}(j\omega)(\mathbf{D}_R^i)^{-1} - \mathbf{G}_{bd,j\omega}^{i+1}.$$

As $\mathbf{G}_{bd,j\omega}^i$ is a feasible solution for (3.32), which achieves $\gamma_1^{i+1} = \gamma_2^i$, the convergence to local optima is proven.

An equally important issue is that of quality of the solution obtained using Algorithm 3.2. Since the approximation problem has multiple local minima and the converged solution depends on the initial value, Algorithm 3.2 can converge to a minima that is worse than using the diagonal blocks. This difficulty is overcome by replacing $\mathbf{G}(j\omega)$ by $\mathbf{G}(j\omega) - \text{diag}(\mathbf{G}_{ii}(j\omega))$ in Algorithm 3.2 and using $\mathbf{G}_{bd,j\omega}^{sub} + \text{diag}(\mathbf{G}_{ii}(j\omega))$ as the locally optimal solution, where $\mathbf{G}_{bd,j\omega}^{sub}$ is the solution obtained using the modified algorithm.

Using the same arguments as used for convergence of the sequence of solutions obtained using Algorithm 3.2, it follows that the modified algorithm always obtains a solution that is at least as good as using the diagonal blocks. Note that replacing $\mathbf{G}(j\omega) - \text{diag}(\mathbf{G}_{ii}(j\omega))$ can bias the algorithm to converge to a local minima close to diagonal blocks. We use a simple approach, where the problem is solved twice using $\mathbf{G}(j\omega)$ and $(\mathbf{G}(j\omega) - \text{diag}(\mathbf{G}_{ii}(j\omega)))$ and select the better solution. It is possible to obtain an improved solution using the available branch and bound methods [108], but this approach is not pursued here with the view of keeping computational requirement low and is a potential area for future research.

3.5.2 Parametric $\mathcal{L}_\infty$ Optimal Identification

In this section, we discuss finding a rational transfer function that explains the frequency response data obtained using Algorithm 3.2. The objective is to find the rational transfer matrix $\tilde{\mathbf{G}}_{bd}(s)$ that best approximates the irrational function and has at least as many unstable poles as $\mathbf{G}(s)$. It would be ideal to directly find $\mathbf{G}_{bd}(s)$ that has the same number of unstable poles as $\mathbf{G}(s)$, but the optimization problem becomes very involved when the number of unstable poles is fixed. In any case, $\mathbf{G}_{bd}(s)$ can be obtained as the optimal Hankel norm approximation of $\tilde{\mathbf{G}}_{bd}(s)$ as discussed in § 3.2.2.

Traditionally, the model identification problem consists of minimizing the least square error or the $\mathcal{H}_2$ norm of $\mathbf{G}_{bd,j\omega_i} - \tilde{\mathbf{G}}_{bd}(j\omega_i)$. In the present case, however, it is more appropriate to instead minimize the worst case error or the $\mathcal{L}_\infty$ norm of $\mathbf{G}_{bd,j\omega_i} - \tilde{\mathbf{G}}_{bd}(j\omega_i)$ (*cf.* (3.28)). In a related context, Helmicki *et al.* [60] formulated the problem of identifying

$\mathcal{H}_\infty$ optimal model from frequency response data for discrete time systems. The same authors have extended their approach to continuous time systems in [61] through a bilinear transformation. The two-step approach of Helmicki *et al.* [60] consists of fitting the frequency response data with finite impulse response (FIR) models followed by Hankel norm approximation, which is similar to the last two steps of the method proposed here. Over the past few years, a number of different approaches have appeared in the literature and the current state of the art of $\mathcal{H}_\infty$ optimal identification can be found in [24].

In this chapter, we parameterize the class of models using transfer functions as compared to the FIR models used by Helmicki *et al.* [61]. An advantage of using the transfer function parametrization is that low order models can be identified directly in the continuous time domain, the disadvantage being that unlike the FIR parametrization, no worst case error bounds are available. Nevertheless, practical experience (particularly in $\mathcal{H}_2$ norm minimization case) suggests that transfer function parametrization works very well. For simplicity, $\tilde{\mathbf{G}}_{bd}(s)$ is identified element by element, where $[\tilde{\mathbf{G}}_{bd}(s)]_{ij}$ is parameterized as:

$$[\tilde{\mathbf{G}}_{bd}(s)]_{ij} = \frac{a(s)}{b(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots b_1 s + b_0}; \quad m \le n$$

In the remaining discussion, we drop the requirement that $\tilde{\mathbf{G}}_{bd}(s)$ has at least as many poles as $\mathbf{G}_{bd}(s)$, as it is easily satisfied by choosing the order of the denominator polynomials sufficiently large. Then, the parameters $a_0 \cdots a_m, b_0 \cdots b_n$, are obtained by solving,

$$\min_{a_0 \cdots a_m, b_0 \cdots b_n} \left| \frac{a(j\omega_k)}{b(j\omega_k)} - [\mathbf{G}_{bd,j\omega_k}]_{ij} \right| \qquad k = 1 \cdots n_\omega \tag{3.33}$$

Note that the objective function in (3.33) is nonlinear, but can be equivalently represented as $|b(j\omega_k)|^{-1}|a(j\omega_k) - b(j\omega_k)[\mathbf{G}_{bd,j\omega_k}]_{ij}|$. Now, we can instead minimize

$$|a(j\omega_k) - b(j\omega_k)[\mathbf{G}_{bd,j\omega_k}]_{ij}| = \sqrt{\mathrm{Re}(a(j\omega_k) - b(j\omega_k)[\mathbf{G}_{bd,j\omega_k}]_{ij})^2 + \mathrm{Im}(a(j\omega_k) - b(j\omega_k)[\mathbf{G}_{bd,j\omega_k}]_{ij})^2}$$

which is easily represented as an LMI problem as follows:

$$\begin{aligned} & \min_{a_0 \cdots a_m, b_0 \cdots b_n \in \mathbf{R}} \gamma_1^2 + \gamma_2^2 \\ \text{subject to} \quad & -\gamma_1^2 \le \mathrm{Re}\left(a(j\omega_k) - b(j\omega_k)[\mathbf{G}_{bd,j\omega_k}]_{ij}\right) \le \gamma_1^2 \\ & -\gamma_2^2 \le \mathrm{Im}\left(a(j\omega_k) - b(j\omega_k)[\mathbf{G}_{bd,j\omega_k}]_{ij}\right) \le \gamma_2^2 \quad k = 1 \cdots n_\omega \end{aligned} \tag{3.34}$$

As $\omega_k \to \infty$, the magnitude of the polynomials $a(j\omega_k), b(j\omega_k)$ becomes unbounded. Thus, the formulation (3.34) inherently emphasizes minimization of γ_1, γ_2 at high frequencies. The following iterative approach can be used, which does not suffers from this limitation:

$$\begin{array}{ll} & \min\limits_{a_0^{(i)}\cdots a_m^{(i)}, b_0^{(i)}\cdots b_n^{(i)} \in \mathbb{R}} \quad \gamma_1^2 + \gamma_2^2 \\ \text{subject to} & -\gamma_1^2 |b^{(i-1)}(j\omega_k)| \le \mathrm{Re}\left(a^{(i)}(j\omega_k) - b^{(i)}(j\omega_k)[\mathbf{G}_{bd,j\omega_k}]_{ij}\right) \le \gamma_1^2 |b^{(i-1)}(j\omega_k)| \\ & -\gamma_1^2 |b^{(i-1)}(j\omega_k)| \le \mathrm{Im}\left(a^{(i)}(j\omega_k) - b^{(i)}(j\omega_k)[\mathbf{G}_{bd,j\omega_k}]_{ij}\right) \le \gamma_2^2 |b^{(i-1)}(j\omega_k)| \\ & b_n = 1 \end{array} \tag{3.35}$$

where $b^{(i-1)}(j\omega_k)$ denotes the identified b polynomial from the previous iteration. In (3.35), the additional constraint $b_n = 1$ is imposed for numerical stability and in general, fixing any one of the unknown parameters suffices. In the $\mathcal{H}_2$ optimal identification literature, methods similar to (3.34) and (3.35) are known as Levi's and Sanathanan and Koerner's method respectively [90]. The sequence of solutions obtained by solving optimization problem (3.35) is not guaranteed to converge, but reasonable solution can be obtained using a few iterations.

3.6 Controller Design

With the availability of $\mathbf{G}_{bd}(s)$ using Algorithm 3.1, the controller design for the modified μ-IM is similar to the conventional μ-IM method. A loop shaping approach can be used to find the stabilizing decentralized controller; however, finding a controller using this method to satisfy (3.16) can be difficult. In this section, we show that with the alternate representation of the μ-IM conditions in terms of $\mathbf{K}_{bd}\mathbf{S}_{bd}(s)$, finding $\mathbf{K}_{bd}(s)$ to satisfy (3.16) reduces to solving a weighted $\mathcal{H}_\infty$ controller design problem for $\mathbf{G}_{bd}(s)$.

Proposition 3.4 Consider that $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ have n_p unstable poles. Let the minimum phase and stable transfer matrix $w(s)$ be chosen such that $|w(j\omega)| = \mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega))$ for all ω. There exists a block diagonal controller $\mathbf{K}_{bd}(s)$ such that $\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) < \mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega))$ for all $\omega \in \mathbb{R}$ iff

$$\underline{\sigma}_H^{-1}(\mathcal{U}(w^{-1}\mathbf{G}_{bd}(s))^*) < 1 \tag{3.36}$$

where $\mathcal{U}(.)$ denotes the unstable part.

Proof: (Sufficiency) Let us define, $\tilde{\mathbf{K}}_{bd}(s) = w(s)\mathbf{K}_{bd}(s)$ and $\tilde{\mathbf{G}}_{bd}(s) = w^{-1}(s)\mathbf{K}_{bd}(s)$. Then, using Proposition 2.3, there exists a $\mathbf{K}_{bd}(s)$ such that,

$$\begin{aligned} \inf_{\mathbf{K}_{bd}(s)} \|w\mathbf{K}_{bd}\mathbf{S}_{bd}(s)\|_\infty &= \inf_{\tilde{\mathbf{K}}_{bd}(s)} \|\tilde{\mathbf{K}}_{bd}(s)(\mathbf{I} + \tilde{\mathbf{G}}_{bd}\tilde{\mathbf{K}}_{bd}(s))^{-1}\|_\infty \\ &= \underline{\sigma}_H^{-1}(\mathcal{U}(w^{-1}\mathbf{G}_{bd}(s))^*) \end{aligned}$$

If (3.36) holds, there exists a $\mathbf{K}_{bd}(s)$ such that

$$\begin{aligned} \|w\mathbf{K}_{bd}\mathbf{S}_{bd}(s)\|_\infty &< 1 \\ \Leftrightarrow \bar{\sigma}(w\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) &< 1 \quad \forall\omega \\ \Leftrightarrow \bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) &< |w(j\omega)|^{-1} \quad \forall\omega \\ \Leftrightarrow \bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) &< \mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega)) \quad \forall\omega \end{aligned} \tag{3.37}$$

where the last inequality holds as $|w(j\omega)| = \mu_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))$ for all ω.

(Necessity) We show the necessity of (3.36) by contradiction. Consider that (3.36) does not hold, but there exists a $\mathbf{K}_{bd}(s)$ such that $\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) < \mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega))\ \forall\omega$. By reversing the series of inequalities used for sufficiency, $\mathbf{K}_{bd}(s)$ must satisfy (3.37). The $\underline{\sigma}_H^{-1}(\mathcal{U}(w^{-1}\mathbf{G}_{bd}(s))^*)$ denotes the least achievable value for $\|w(s)\mathbf{K}_{bd}\mathbf{S}_{bd}(s)\|_\infty$ for all LTI controllers. Then, $\|w\mathbf{K}_{bd}\mathbf{S}_{bd}(s)\|_\infty$ being less than 1, despite $\underline{\sigma}_H^{-1}(\mathcal{U}(w^{-1}\mathbf{G}_{bd}(s))^*)$ being equal to or greater than 1 is a contradiction and the necessity of (3.36) follows. ■

In Proposition 3.4, we assumed that $w(s)$ is stable and minimum phase. In general, $w(s)$ can have RHP zeros and RHP poles at same the location as $\mathbf{G}_{bd}(s)$. Allowing $w(s)$ to be unstable or non-minimum phase provides no advantage, as following the discussion in § 2.3, we can simply replace $w(s)$ by its minimum and stable part in (3.36). On relaxing this assumption, however, $w(s)$ that achieves $|w(j\omega)| = \mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega))$ becomes non-unique, where the different instances of $w(s)$ are related by a unitary transformation.

Proposition 3.4 effectively reduces the task of finding a block decentralized controller to satisfy μ-IM condition (3.16) to finding the minimum phase and stable $w(s)$ such that $|w(j\omega)| = \mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega))$ and (3.36) holds. When (3.36) is satisfied, the standard $\mathcal{H}_\infty$ optimal control design techniques can be used to find the stabilizing decentralized controller.

Remark 3.5 In practice, it can be difficult to find $w(s)$ that satisfies $|w(j\omega)| = \mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega))$ for all $\omega \in \mathbb{R}$. This difficulty can be overcome by recognizing that any $w(s)$ that lower bounds $\mu_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))$ at all frequencies, if (3.36) holds,

$$\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) < |w(j\omega)|^{-1} \quad \Rightarrow \quad \bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) < \mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega))$$

Thus, for a given $\mathbf{G}_{bd}(s)$ the existence of a decentralized stabilized controller can be established by verifying (3.36) with $w(s)$ that lower bounds $\mu_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))$.

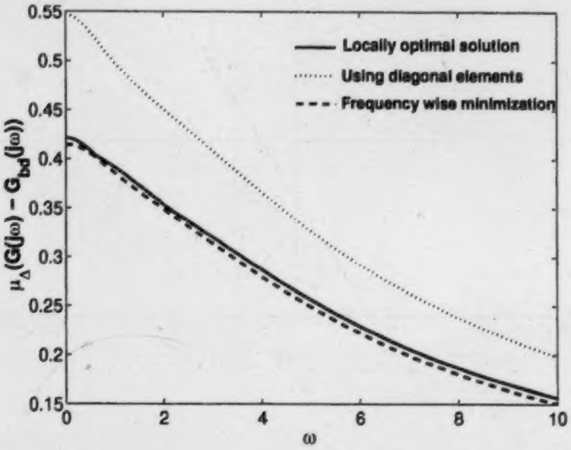


Figure 3.4: Efficiency of proposed method for optimal block diagonal approximation

3.7 Numerical Example

In this section, we demonstrate the efficiency of Algorithm 3.1 for obtaining optimal block diagonal approximation and the controller design method discussed in the previous sections using a simple example.

Consider the following system:

$$\mathbf{G}(s) = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & \beta_1 & \beta_1 \\ 0 & 2 & 0 & 0 & \beta_1 & 1 & \beta_1 \\ 0 & 0 & 3 & 0 & \beta_1 & \beta_1 & 1 \\ 0 & 0 & 0 & -4 & 1 & 0.4 & 0.4 \\ \hline 1 & \beta_2 & \beta_2 & 1 & 0 & 0 & 0 \\ \beta_2 & 1 & \beta_2 & 0.6 & 0 & 0 & 0 \\ \beta_2 & \beta_2 & 1 & 0.6 & 0 & 0 & 0 \end{array}\right]; \quad \beta_1 = 0.5, \beta_2 = 0.1$$

A set of equally spaced frequencies in the range $0-10$ is chosen and the locally optimal diagonal approximation is obtained using the following steps:

- Algorithm 3.2 is used for frequency-wise minimization. The algorithm achieves 3 decimal digits of accuracy as compared to the locally optimal solution in 2 iterations.

- We fit 4^{th} or lower order models for the frequency data using the formulation (3.35) with 2 iterations.

- The identified model has 5 unstable poles, which is reduced to a model with 3 unstable poles using the Hankel norm approximation method discussed in § 3.2.2.

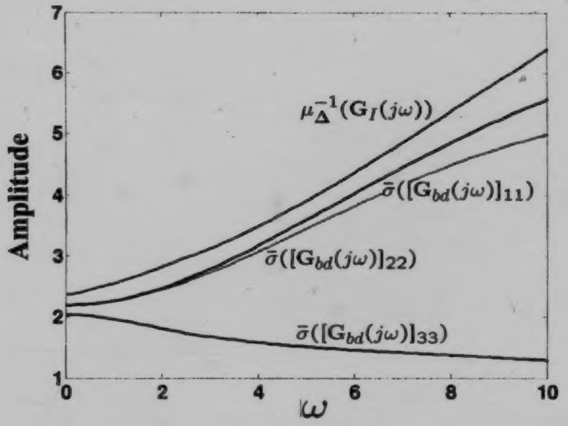


Figure 3.5: Validation of modified μ-IM for stabilizing decentralized controller designed using independent designs

The $\mathbf{G}_{bd}^{sub}(s)$, as obtained following these steps, is given as:

$$\text{diag}\left(\frac{-0.002s^2+2.22s+3.421}{s^2+2.92s-3.96}, \frac{-0.01045s^2+2.04s+5.98}{s^2+2.53s-9.69}, \frac{-0.01575s^2+1.842s+4.98}{s^2+1.77s-8.99} \right)$$

For comparison purposes, we also calculate the sub-optimal solution $\mathbf{G}_{bd}^{diag}(s)$ by reducing the order of diagonal elements of $\mathbf{G}$. In this case, 5 Hankel singular values of the stable part of $\mathbf{G}_{bd}^{diag}(s)$ are negligible, which are removed to get a reduced order model given as:

$$\text{diag}\left(\frac{2.075s+3.272}{s^2+2.96s-4.16}, \frac{1.33s+3.896}{s^2+2.06s-7.762}, \frac{-0.006s^2+1.255s+3.533}{s^2+1.422s-10.31} \right)$$

To show the advantage of Algorithm 3.1 over using diagonal elements, $\gamma^{sub} = \mu_{\Delta}(\mathbf{G}(j\omega) - \mathbf{G}_{bd}^{sub}(j\omega))$ and $\gamma^{diag} = \mu_{\Delta}(\mathbf{G}(j\omega) - \mathbf{G}_{bd}^{diag}(j\omega))$ are compared in Figure 3.4. The relative difference between γ^{diag} and γ^{sub} is 0.23 at the zero frequency, which monotonically reduces to 0.13 for $\omega = 10$. This significant reduction in the approximation error is useful for finding the stabilizing controller easily. Figure 3.4 also shows that the γ^{sub} closely matches the approximation error obtained using frequency wise minimization. Thus, (at least for this example), the conservativeness in using the two-step approach for identifying a model, with same number of unstable poles as the system, is minimal.

Next, we consider the controller design part. For the locally optimal diagonal approximation, the following weight approximates $\mu_{\Delta}(\mathbf{G}_I(j\omega))$ closely,

$$w(s) = \frac{0.107s^2 + 2.12s + 10.54}{s^2 + 2.06s + 7.762}$$

Using this $w(s)$, $\underline{\sigma}_H(\mathcal{U}(w^{-1}\mathbf{G}_{bd}^{sub}(s))^*) = 1.06 > 1$ and standard $\mathcal{H}_\infty$ optimal controller design technique is used to find a decentralized stabilizing controller. The plots of $\mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega))$ and $\bar{\sigma}([\mathbf{G}_{bd}(j\omega)]_{ii})$, $i = 1, 2, 3$ are shown in Figure 3.5, where $\mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega)) > \bar{\sigma}([\mathbf{G}_{bd}(j\omega)]_{ii})$, as expected. On the other hand, for the suboptimal solution obtained using the diagonal elements, $\underline{\sigma}_H(\mathcal{U}(w^{-1}(s)\mathbf{G}_{bd}^{diag}(s))^*) = 0.524 < 1$. Then, the conservativeness of using the diagonal elements to find a suboptimal solution is emphasized.

3.8 Chapter Summary

In this chapter, we extended the practical applicability of μ-IM to unstable systems. The decentralized controller is designed based on a block diagonal approximation that is different from the block diagonal elements, but has same number of unstable poles as the system. By expressing the μ-IM in terms of transfer matrix from disturbances to inputs, it is shown that:

- The block diagonal approximation can be (sub-optimally) chosen by minimizing the scaled $\mathcal{L}_\infty$ distance between the system and the approximation.
- The task of designing the controller based on the block diagonal approximation can be reduced to solving a weighted $\mathcal{H}_\infty$ optimal controller design problem.
- The decentralized stabilizing controller inherently minimizes an upper bound on the input requirement for stabilization, but the bound is very loose.

We have shown that when the system is partitioned into 2 blocks, the optimal block diagonal approximation can be obtained by order reduction of diagonal blocks. For the general case, a step-wise numerical approach is presented for finding the locally optimal solution to the block diagonal approximation problem. The proposed approach involves solving the approximation problem at a set of frequencies followed by $\mathcal{L}_\infty$ optimal identification.

One promising approach for identifying low order continuous models from frequency response data is to use the Nevanlinna-Pick interpolation theory [6]. The interpolation theory parameterizes all rational stable functions that can pass through the given set of (adjusted) complex valued data. This method has been used by Chen *et al.* [25] for $\mathcal{H}_\infty$ optimal identification and can easily be extended to the $\mathcal{L}_\infty$ case. The present difficulties in using this approach are (i) the order of the model is the same as the number of data points; and more importantly, (ii) due to an over-emphasis on the given set of frequencies, the interpolating function shows inter-sample oscillations.

The primary limitation of choosing the block diagonal approximation by minimizing the scaled $\mathcal{L}_\infty$ distance is that the properties of the approximant are not taken into account. As shown in this chapter, whether the stabilizing controller can be easily found depends on the minimum Hankel singular value of the approximation. A better approach is to use a multi-objective optimization framework, where the $\mathcal{L}_\infty$ distance between the system and the approximation is minimized and simultaneously the minimum Hankel singular of the approximation is maximized. This non-trivial problem is a topic for future work.

Chapter 4

Block Relative Gain: Properties and Pairing Rules

Block relative gain (BRG) is a useful method for finding suitable pairings for block decentralized control. In this chapter, we present some new algebraic properties of BRG and establish its relation with closed loop stability, controllability, block diagonal dominance and interactions. We show that the common conjecture that a system is weakly interacting, if BRG is close to the Identity matrix, is not true. Based on these insights, simple rules for pairing of variables are proposed. We also extend the known method for calculating RGA for interacting systems to BRG.[1]

4.1 Introduction

Decentralized controllers are widely used in the process industries due to their simplicity. The performance of a fully decentralized controller can be poor in presence of severe process interactions. In such situations, the use of full multivariable controller is an attractive alternative. On the other hand, decentralized controllers are easier to design, tune and can be made fault-tolerant more easily as compared to full multivariable controllers [18]. An alternative to either fully decentralized or a full multivariable controller is the use of block decentralized controller, which has a structure in between the two extremes. Block decentralized controllers balance the high performance given by full multivariable controllers and the easier implementation and maintenance associated with fully decentralized controllers. The use of block decentralized controllers can be further justified by the fact that in most industrial processes, the interactions are limited in scope and do not include the full scope of the plant. Thus, these processes are most suitably controlled in a *block-wise* fashion. Since the number of such blocks and block decentralized alternatives increases rapidly with system size, controller design for every alternative is impractical at design stage. Thus, effective tools are required to get an estimate of closed loop properties without designing the actual controller. In this chapter, we present results concerning such a tool, *i.e.*, Block relative gain (BRG) [83].

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[1] This work has been published in Industrial Engineering & Chemistry Research [72] and a shorter version in the proceedings of ADCHEM 2003, Hong Kong, P.R. China [73].

The BRG generalizes the concept of the relative gain array (RGA) [17] to block pairings. It is a powerful technique for input-output controllability analysis and screening alternatives quickly for block decentralized control at the design stage. The development of the BRG is based on the assumption of perfect control[2]. Arkun [3] has argued that rigorous closed loop stability and performance analysis is not possible under this assumption and has suggested the use of the dynamic block relative gain. The BRG has also been extended to handle non-square [93] and non-linear [82] cases. However, the applicability of these extensions of the BRG is limited due to their dependence upon controller tuning and their extensive computational requirements. These approaches are not considered here and the discussion is limited to square, linear time invariant (LTI) and stable systems, unless otherwise stated.

During the past few decades, the RGA has been studied extensively [51, 65, 71, 113] and its properties are well understood, but the BRG has largely been overlooked. Some researchers [20, 85] have found relations between the BRG and Euclidian condition number. It is shown that generally, a system is difficult to control, if the maximum singular value of BRG is large. Chen *et al.* [23] have further considered the role of the BRG in robustness analysis. Despite these studies, contrary to the RGA, BRG has not gained widespread popularity and block pairings are selected primarily based on heuristics [19, 29]. The use of heuristics can be attributed to lack of a study showing that similar to the RGA, information regarding closed loop properties can be obtained using BRG. This motivates the present work.

In this chapter, we present some novel algebraic properties of BRG. We establish the connection between BRG and closed loop properties like stability, controllability, block diagonal dominance and interactions. Manousiouthakis *et al.* [83] have claimed that a system is *weakly* interacting, if BRG is close to the Identity matrix and have proposed a pairing algorithm based on this statement. We show that this claim is incorrect. Further, a system can have large interactions despite BRG being exactly the Identity matrix. It should be emphasized, however, that if the singular values of BRG are very different from unity, the closed loop system has large interactions. Based on these insights, simple rules for variable pairing are proposed. Simultaneously, we present a note on calculation of BRG, when the system contains integrating elements.

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[2] Perfect control is achieved, when the output, $y(t)$ is equal to the reference, $r(t)\ \forall t > 0$ [102].

Grosdidier *et al.* [51] pointed out that modelling uncertainties and changing operating conditions make it very difficult to develop reliable dynamic models for chemical processes and often, only steady state gain information is available. With this motivation, we focus on extracting useful feedback properties from gain information, though most of the results presented are directly generalizable to higher frequencies.

The organization of this chapter is as follows. In §4.2, we revisit the development of BRG and cite the limitations of existing pairing rules; in §4.3, we present some algebraic properties of BRG; the main results of this chapter are contained in §4.4, where it is shown that BRG can be used to assess some desired closed loop properties; in §4.5, alternate pairing rules are proposed and illustrative examples are presented; in §4.6, we consider the case, when the system matrix contains integrating elements and §4.7 concludes this chapter.

4.2 Preliminaries

In this section, we introduce the concepts of relative gain and BRG. We present the BRG based pairing rules due to Manousiouthakis [83] and point to their limitations.

As before, the transfer function matrix relating outputs and inputs of the system is represented as $\mathbf{G}(s)$ in this chapter. The steady state gain matrix is represented as $\mathbf{G}(0)$ or simply $\mathbf{G} \in \mathbb{R}^{n\times n}$. The objective is to decompose the original system into a set of M non-overlapping square subsystems such that, $\mathbf{G}_{ii} \in \mathbb{R}^{m_i\times m_i}$; $i = 1, 2 \cdots M$, $\sum_i m_i = n$. $\mathbf{G}_{ij} \in \mathbb{R}^{m_i\times m_j}$ represents the ij^{th} block of $\mathbf{G}$ or the block gain between $\mathbf{y}_i$ and $\mathbf{u}_j$. The pair $(\mathbf{y}_i, \mathbf{u}_j)$ denotes the variables related by $\mathbf{G}_{ij}(s)$.

Definition 4.1 *Relative gain* [17] for variable pairing $(\mathbf{y}_i, \mathbf{u}_j)$ is defined as the ratio of two gains representing first the process gain in an isolated loop and second, the apparent process gain in the same loop when all other control loops are closed,

$$\begin{aligned} \lambda_{ij} &= g_{ij}\left[\mathbf{G}(0)^{-1}\right]_{ji} \\ \mathbf{\Lambda}(\mathbf{G}) = [\lambda_{ij}] &= \mathbf{G}(0)\circ\mathbf{G}(0)^{-T} \end{aligned}$$

where $\circ$ is the Hadamard product and $\mathbf{G}(0)^{-T}$ is transpose of the inverse of $\mathbf{G}(0)$. $\mathbf{\Lambda}(\mathbf{G})$ is called RGA. Manousiouthakis *et al.* [83] extended the concept of the RGA to BRG for synthesizing block decentralized controllers.

Definition 4.2 *Block relative gain* [83] for variable pairing $(\mathbf{y}_1, \mathbf{u}_1)$ is defined as the ratio of the open loop block gain and apparent block gain in the same loop when all other control loops are closed,

$$[\mathbf{\Lambda}_B(s)]_{11} = \mathbf{G}_{11}(s)[\mathbf{G}^{-1}(s)]_{11} \tag{4.1}$$

where $\mathbf{G}_{11}(s)$ is the $m_1 \times m_1$ transfer function matrix, $m_1 \leq n$, relating the first m_1 inputs and outputs of $\mathbf{G}$ and $[\mathbf{G}^{-1}(s)]_{11}$ is the corresponding block of $\mathbf{G}^{-1}(s)$.

Precisely, (4.1) represents the expression for left-BRG. Similarly, right-BRG can be calculated as $[\mathbf{G}^{-1}(s)]_{11}\mathbf{G}_{11}(s)$. Since the left and right-BRG share common properties, consideration of right-BRG is omitted from this discussion.

4.2.1 BRG Revisited

Let the LTI system, $\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s) + \mathbf{d}(s)$ be conformably partitioned such that $\mathbf{G}_{11}(s)$ is an $m_1 \times m_1$ transfer function matrix,

$$\begin{aligned} \mathbf{y_1}(s) &= \mathbf{G}_{11}(s)\mathbf{u_1}(s) + \mathbf{G}_{12}(s)\mathbf{u_2}(s) + \mathbf{d}(s) \\ \mathbf{y_2}(s) &= \mathbf{G}_{21}(s)\mathbf{u_1}(s) + \mathbf{G}_{22}(s)\mathbf{u_2}(s) + \mathbf{d}(s) \end{aligned} \tag{4.2}$$

When $(\mathbf{y_2}, \mathbf{u_2})$ is perfectly controlled and $\mathbf{d}(s) \approx 0$, at steady state, $(\mathbf{y_1}, \mathbf{u_1})$ are related through the Schur complement of $\mathbf{G}_{22}$ in $\mathbf{G}$ [102],

$$\mathbf{y_1} = \bar{\mathbf{G}}_{11}\mathbf{u_1}; \quad \bar{\mathbf{G}}_{11} = \mathbf{G}_{11} - \mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21} \tag{4.3}$$

In (4.3), it is assumed that the subsystem $\mathbf{G}_{22}$ is nonsingular, though it is not necessary for existence of the BRG, as is shown later. For partitioned matrices [62], $[\mathbf{G}^{-1}]_{11} = \bar{\mathbf{G}}_{11}^{-1}$. Now, the steady state block relative gain between $(\mathbf{y}_1, \mathbf{u}_1)$ can be defined as,

$$[\mathbf{\Lambda}_B]_{11} = \mathbf{G}_{11}[\mathbf{G}^{-1}]_{11} \tag{4.4}$$

Similarly, $\mathbf{G}$ can also be partitioned into M diagonal and conformal off-diagonal blocks, such that $\mathbf{G}_{ii} \in \mathbb{R}^{m_i \times m_i}$; $i = 1, 2 \cdots M$. Then,

$$[\mathbf{\Lambda}_B]_{ii} = \mathbf{G}_{ii}[\mathbf{G}^{-1}]_{ii} \tag{4.5}$$

Manousiouthakis *et al.* [83] have suggested choosing the pairings such that the eigenvalues of all the corresponding BRGs are close to 1. This pairing rule is based on the conjecture that a system is *weakly* interacting, if the BRG is close to the Identity matrix, and is similar to the pairing rules for RGA prevalent then. Now, it is well known that pairing on RGA elements close to 1 can lead to pairings with significant interactions. Since relative gains are a special case of BRG, the utility of this rule is also limited for choosing block pairings.

4.3 Algebraic Properties

In this section, we present some new properties of the BRG and alternative proof of a previously known property. Some of the properties are used to establish some more important properties of BRG in later sections, while others are of purely algebraic interest.

Property 4.1 The individual elements of $[\mathbf{\Lambda}_B]_{11} \in \mathbb{R}^{m_1 \times m_1}$, β_{ij} can be alternatively computed as the weighted sum of RGA elements as,

$$\beta_{ij} = \sum_{k=1}^{m_1} \frac{g_{ik}}{g_{jk}} \lambda_{jk} \tag{4.6}$$

Proof: From (4.4),

$$\beta_{ij} = \frac{\sum_{k=1}^{m_1} (-1)^{j+k} g_{ik} \det(\mathbf{G}^{jk})}{\det(\mathbf{G})} \tag{4.7}$$

where $\mathbf{G}^{jk}$ is system $\mathbf{G}$ with j^{th} row and k^{th} column deleted. Yu and Luyben [113] have shown that,

$$\frac{(-1)^{j+k} \det(\mathbf{G}^{jk})}{\det(\mathbf{G})} = \frac{\lambda_{jk}}{g_{jk}} \tag{4.8}$$

Now, (4.6) can be obtained by substituting (4.8) into (4.7). ■

A special case of (4.6) is seen for diagonal elements of $[\mathbf{\Lambda}_B]_{11}$, $(i = j)$, [83]

$$\beta_{ii} = \sum_{k=1}^{m_1} \lambda_{ik}$$

This property can be helpful in reducing computational load, when the BRG is to be calculated for different decompositions of large systems.

It is known that the row and column sum of the RGA is equal to 1 [17]. In order to extend this property to BRG, we define $\mathcal{I}$ as the ensemble of the m_1-dimensional ordered index sets chosen from the first n natural numbers. For example, for $n = 3$ and $m_1 = 2$, $\mathcal{I}$ has the following elements: $(1, 2)$, $(1, 3)$ and $(2, 3)$. Given a matrix $\mathbf{A}$, every $p, q \in \mathcal{I}$, define a submatrix, denoted as $\mathbf{A}_{pq}$, made up of rows and columns of $\mathbf{A}$ indexed by p and q respectively.

Property 4.2 Let $p, q \subset \mathcal{I}$. Then $\mathbf{y}_p \subset \mathbf{y}$, $\mathbf{u}_q \subset \mathbf{u}$ and $[\mathbf{\Lambda}_B(\mathbf{G}_{pq})]_{11}$ is the BRG between $\mathbf{y}_p$ and $\mathbf{u}_q$. Then,

$$\sum_{q \subset \mathcal{I}} [\mathbf{\Lambda}_B(\mathbf{G}_{pq})]_{11} = \frac{m_1}{n} \frac{n!}{m_1!(n-m_1)!} \cdot \mathbf{I}_{m_1} \quad \forall p \subset \mathcal{I}$$

Proof: Since $[\mathbf{\Lambda}_B(\mathbf{G}_{pq})]_{11} = \mathbf{G}_{pq}[\mathbf{G}^{-1}]_{qp}$, for any $p, q \subset \mathcal{I}$, summation over all $q \subset \mathcal{I}$ yields

$$\sum_{q\subset\mathcal{I}} [\mathbf{\Lambda}_B(\mathbf{G}_{pq})]_{11} = \sum_{q\subset\mathcal{I}} \mathbf{G}_{pq}[\mathbf{G}^{-1}]_{qp} = \sum_{k=1}^{\mathrm{s}_q} \mathbf{G}_{pk}[\mathbf{G}^{-1}]_{kp}$$

By construction, the cardinality of the set $\bigcup q$ for all $q \in \mathcal{I}$ is $(m_1 \cdot n!)/(m_1! \cdot (n-m_1)!)$, where the set $\bigcup q$ contains the first n natural numbers repeated $(m_1 \cdot n!)/(n \cdot m_1! \cdot (n-m_1)!)$ times. Now the result follows by noting that,

$$\sum_{k=1}^{\mathrm{s}_q} \mathbf{G}_{pk}[\mathbf{G}^{-1}]_{kp} = \frac{m_1}{n} \frac{n!}{m_1!(n-m_1)!} \cdot \mathbf{I}_{m_1}$$

■

A similar result can be obtained by summing right-BRG over all $p \subset \mathcal{I}$ and for any $q \subset \mathcal{I}$. An interesting property of BRG is seen for the case when $m_1 = m_2 = \cdots m_M = m$. Then m is an exact divisor of n. Let q_i be defined as

$$q_i = \{q_i \subset \mathcal{I} \mid q_i \bigcap q_j = \emptyset, \ \bigcup_i q_j = \{1, 2 \cdots n\}\} \ \ \forall i, j = 1 \cdots M, i \neq j$$

Then the following relation holds,

$$\sum_{q_i\subset\mathcal{I}} [\mathbf{\Lambda}_B(\mathbf{G}_{pq_i})] = \mathbf{I}_m \ \ \forall p \subset \mathcal{I}$$

Essentially q_i's partition the input set into smaller sets of equal dimension. For $m = 1$, this result reduces to the known result for RGA.

Property 4.3 Let the gain matrix, $\mathbf{G}$ be scaled as $\mathbf{G}^s = \mathbf{S}_1\mathbf{G}\mathbf{S}_2$. $\mathbf{S}_1 = \mathrm{diag}(s_{1i})$ and $\mathbf{S}_2 = \mathrm{diag}(s_{2i})$, $i = 1 \cdots n$, are output and input scaling matrices respectively and are real. If $\mathbf{S}_1$ and $\mathbf{S}_2$ are partitioned such that $\mathbf{S}_1 = \mathrm{diag}(\mathbf{S}_{11}, \mathbf{S}_{12})$, $\mathbf{S}_2 = \mathrm{diag}(\mathbf{S}_{21}, \mathbf{S}_{22})$ and $\mathbf{S}_{11}, \mathbf{S}_{21} \in \mathbb{R}^{m_1 \times m_1}$, then [83],

$$[\mathbf{\Lambda}_B^s]_{11} = \mathbf{S}_{11}\,[\mathbf{\Lambda}_B]_{11}\,\mathbf{S}_{11}^{-1} \tag{4.9}$$

Proof: Using (4.6) and noting that λ_{ij} is independent of scaling [17],

$$\beta_{ij}^s \ = \ \sum_{k=1}^{m_1} \frac{(s_{1i}g_{ik}s_{2k})}{(s_{1j}g_{jk}s_{2k})}\lambda_{jk} \ = \ \sum_{k=1}^{m_1} \frac{(s_{1i}g_{ik})}{(s_{1j}g_{jk})}\lambda_{jk} \ = \ s_{1i}\beta_{ij}\frac{1}{s_{1j}} \tag{4.10}$$

Recognizing that $\mathbf{S}_{11}^{-1} = diag\{1/s_{11}, 1/s_{12} \cdots 1/s_{1m}\}$, the equivalence of (4.9) and (4.10) can be shown. ■

Based on (4.9) and (4.10), the following observations are made:

(i) β_{ij}^{s} is independent of input scaling, but dependent on output scaling.

(ii) $[\Lambda_B]_{ii}$ is independent of scaling of $[\Lambda_B]_{jj}$, for all $i, j = 1, 2 \cdots M, j \neq i$.

(iii) If all outputs are scaled by the same scalar, then $\beta_{ij}^{s} = \beta_{ij}$. This can be shown by setting $s_{11} = s_{12} = \cdots = s_{1m}$ in (4.10).

(iv) The diagonal elements of BRG, β_{ij}, $(i = j)$, are independent of scaling.

In the development of the BRG, it was assumed that $\mathbf{G}_{ii}$ is non-singular. The next property shows that the existence of $[\Lambda_B]_{ii}$ does not depend on the fulfillment of this assumption.

Property 4.4 If $\mathbf{G}$ is non-singular and $\mathbf{G}_{ii}$ is singular, then $[\Lambda_B]_{ii}$ exists and is singular.

Proof: Since $\mathbf{G}$ is non-singular by assumption, $\mathbf{G}^{-1}$ and thus $[\mathbf{G}^{-1}]_{ii}$ exists. Thus, $[\Lambda_B]_{ii}$ exists, but is rank deficient due to rank deficiency of $\mathbf{G}_{ii}$ (*cf.* (4.5)). ■

Example 4.1 Consider the gain matrix $\mathbf{G}$ decomposed into 2 × 2 and 1 × 1 blocks,

$$\mathbf{G} = \begin{pmatrix} 1 & 2 & 1.5 \\ 1 & 2 & 4 \\ 3 & 1 & 5 \end{pmatrix}; \quad [\Lambda_B]_{11} = \begin{pmatrix} 1.6 & -0.6 \\ 1.6 & -0.6 \end{pmatrix}; \quad [\Lambda_B]_{22} = 0$$

Clearly, the first 2 × 2 block of $\mathbf{G}$ is singular. For $((y_1\text{-}y_2, u_1\text{-}u_2), (y_3, u_3))$ pairings, $[\Lambda_B]_{11}$ exists, but is singular.

Property 4.5 For some specified partitioning of the system,

(i) $\mathbf{G}$ being block triangular implies that the corresponding $[\Lambda_B]_{ii} = \mathbf{I}_{m_i}$ for all $i = 1 \cdots M$.

(ii) $[\Lambda_B]_{ii} = \mathbf{I}_{m_i}$ for all $i = 1 \cdots M$ does not imply that $\mathbf{G}$ is block triangular.

Proof: (i) For block triangular matrices, $[\mathbf{G}^{-1}]_{ii} = [\mathbf{G}_{ii}]^{-1}$. Then, using (4.5), $[\Lambda_B]_{ii} = \mathbf{G}_{ii}[\mathbf{G}_{ii}]^{-1} = \mathbf{I}_{m_i}$.

(ii) When only SISO pairings are used, the BRGs are the same as the diagonal elements of RGA and the converse is proved trivially. To show that it is true for any arbitrary partitioning, it would suffice to construct an example showing that $[\Lambda_B]_{ii}$ can be the Identity matrix for all i even when $\mathbf{G}$ is not block triangular. Let the system be partitioned in accordance to (4.2). Using (4.3) and (4.4),

$$[\Lambda_B]_{11} = [\mathbf{I} - \mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}]^{-1}$$

If one of the pairs, $\{\mathbf{G}_{12}, \mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}\}$ and $\{\mathbf{G}_{12}\mathbf{G}_{22}^{-1}, \mathbf{G}_{21}\mathbf{G}_{11}^{-1}\}$ lie in null space of each other, $[\mathbf{\Lambda}_B]_{11}$ is the Identity matrix. In either case, $\mathbf{G}_{12}$ is singular. Similarly, if one of the pairs, $\{\mathbf{G}_{21}\ , \mathbf{G}_{11}^{-1}\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\}$ and $\{\mathbf{G}_{21}\mathbf{G}_{11}^{-1}\ , \mathbf{G}_{12}\mathbf{G}_{22}^{-1}\}$ lie in null space of each other, $[\mathbf{\Lambda}_B]_{22}$ is the Identity matrix. Then, $\mathbf{G}_{21}$ is singular. Clearly, it is not required that one or both of $\mathbf{G}_{12}$ and $\mathbf{G}_{21}$ be zero matrices or $\mathbf{G}$ be block triangular for $[\mathbf{\Lambda}_B]_{11} = \mathbf{I}_{m_1}$ and $[\mathbf{\Lambda}_B]_{22} = \mathbf{I}_{m_2}$. Similar arguments can be used to reach this conclusion, when the system is to be partitioned into any arbitrary number of blocks. ■

Example 4.2 Consider the system

$$\mathbf{G} = \begin{pmatrix} 0.2 & 2 & 2.5 & 1.1 \\ 1.5 & 0.4 & 2.5 & 1.1 \\ 1.3 & -1.6 & 0.5 & 1 \\ -1.3 & 1.6 & 2 & 0.1 \end{pmatrix}.$$

For a 2 × 2 and 2 × 2 decomposition of $\mathbf{G}$, $[\mathbf{\Lambda}_B]_{11}$ and $[\mathbf{\Lambda}_B]_{22}$ are equal to the Identity matrix, despite the system not being block triangular. Note that in this example, both the off-diagonal blocks are singular.

If $\mathbf{G}$ is block triangular, then the system is one-way interacting. In this case, the stability of the individual loops implies the stability of the overall system. Property 4.5 shows that this cannot be inferred directly from BRG. In the context of SISO pairings, this property relates to the diagonal elements of RGA only. Some researchers, *e.g.* Hovd and Skogestad [65] (also see [102, Theorem 10.3]), have claimed that RGA being the Identity matrix implies that the system is triangular or can be permuted to the triangular form. By means of a counterexample, Johnson and Shapiro [71] have shown that for $\mathbf{G} \in \mathbb{R}^{n \times n}, n \geq 4$, this is not true. Whereas the example in [71] is purely mathematical, Braatz *et al.* [12] found that the RGA can be arbitrary close to the Identity matrix for real industrial processes, which are neither triangular or can be permuted to the triangular form.

Property 4.6 If the rows and columns of the gain matrix, $\mathbf{G}$ are permuted such that, $\mathbf{G}^p = \mathbf{P}_1\mathbf{G}\mathbf{P}_2$, where $\mathbf{P}_1$ and $\mathbf{P}_2$ are permutation matrices, and further, if $\mathbf{P}_1$ and $\mathbf{P}_2$ can be partitioned as,

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{P}_{11} & 0 \\ 0 & \mathbf{P}_{22} \end{bmatrix} \quad \mathbf{P}_2 = \begin{bmatrix} \mathbf{P}_{21} & 0 \\ 0 & \mathbf{P}_{22} \end{bmatrix}$$

then BRG for the permuted system is,

$$[\mathbf{\Lambda}_B^p]_{11} = \mathbf{P}_{11}\,[\mathbf{\Lambda}_B]_{11}\,\mathbf{P}_{11}^{-1} \tag{4.11}$$

Proof: See [83]. ■

It should be noted that $[\mathbf{\Lambda}_B^p]_{11}$ and $[\mathbf{\Lambda}_B]_{11}$ are equivalent from a variable pairing point of view, as either of these represent the block gain between the same set of inputs and outputs. Then, we may naturally seek the the total number of *distinct* block decentralized alternatives for a given system. This issue is addressed next by using the concept of partition functions, but before that a formal definition of partition function is necessary.

Definition 4.3 A partition of a positive integer n is a finite non-increasing sequence of positive integers $q_1, q_2, \cdots q_r$, such that $\sum_{i=1}^{r} q_i = n$. The *partition function* $p(n)$ is the number of possible partitions of n [2].

Essentially, $p(n)$ represents the number of ways of writing n as sum of smaller integers, where the order of the addends is not considered significant. In the present context, $p(n)$ represents the number of ways of block partitioning the given system.

Property 4.7 The number of distinct block decentralized alternatives, $N(n)$ for a square system is given by,

$$N(n) = \sum_{i=1}^{p(n)} \frac{(n!)^2}{\prod_{j=1}^{M} (m_{j,i}!)^2 a_{j,i}!}; \quad \sum_{k=1}^{M} m_{k,i} = n \tag{4.12}$$

where a_j is the number of occurrences of j in the sequence $\{m_1, m_2 \cdots m_M\}$.

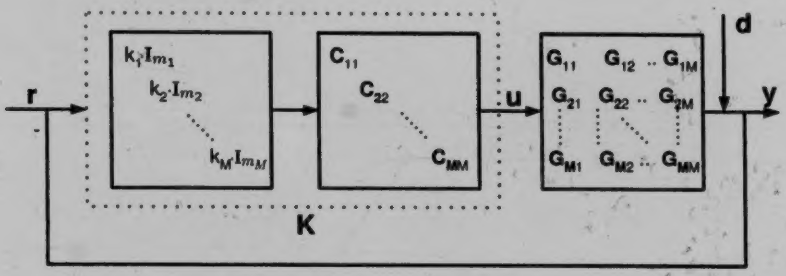
Proof: For any given decomposition, the total number of ways, in which n outputs and n inputs can be permuted is $n! \times n! = (n!)^2$. Considering that permutation within a block gives rise to equivalent BRGs, the total number of distinct permutations decreases to,

$$\left(\frac{n!}{m_1! m_2! \cdots m_M!}\right)^2; \quad \sum_{k=1}^{M} m_k = n$$

Let there exist i, j such that $m_i = m_j$, $i, j \leq M$. Then, the cases where the same set of outputs and inputs are assigned to i^{th} or j^{th} block are the same. Let, a_j represent the number of occurrences of j in the sequence $\{m_1, m_2 \cdots m_M\}$. Thus, the total number of distinct alternatives is given as,

$$\frac{(n!)^2}{\prod_{j=1}^{M} (m_j!)^2 a_j!}; \quad \sum_{k=1}^{M} m_k = n \tag{4.13}$$

Now, $p(n)$ represents the total number of such possible decompositions (including the fully centralized case). Thus, an expression for $N(n)$ is realized by summing (4.13) over $p(n)$. ■


r
-
$k_1 I_{m_1}$
$k_2 I_{m_2}$
$k_M I_{m_M}$
C_{11}
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K
u
G_{11} G_{12} .. G_{1M}
G_{21} G_{22} .. G_{2M}
G_{M1} G_{M2} .. G_{MM}
d
y

Figure 4.1: Closed loop system with integral action controller

Note that the above expression for $N(n)$ also includes the *fully centralized* case. For the *fully decentralized* case, $m_j = 1$ for all j, $a_1 = n$ and $a_j = 0$ for all $j > 1$. Therefore the number of alternatives is simply $n!$ (*cf.* (4.13)). Development of an analytical expression for $N(n)$ explicitly in terms of n is beyond the scope of this thesis. $N(n)$ for some typical values of n is presented in Table 4.1. By evaluating $N(n)$ for different values of n, $n \leq 40$, the following empirical relation can be obtained,

$$N(n) \approx n!^{1.52} \tag{4.14}$$

n	$P(n)$	$n!$	$N(n)$
3	3	6	16
4	5	24	131
5	7	120	1496
6	11	720	22482
8	22	40320	9934563
10	42	3628800	9.0852×10^{9}
15	176	1.3077×10^{12}	2.5273×10^{18}

Table 4.1: $N(n)$ for some typical values of n

In many practical situations, the maximum number of blocks or the maximum dimension of individual blocks is constrained. Such cases can be handled using the concept of restricted partitions. Andrews [2] provides a detailed discussion of partition theory.

4.4 Closed Loop Properties

Throughout this section, we assume that the controller has integral action to give asymptotically zero tracking error. Then, the controller $\mathbf{K}_{ii}(s)$, can be expressed as

$(k_i \cdot \mathbf{I}_{m_i}/s)\mathbf{C}_{ii}(s)$, $k_i > 0$ (see Figure 4.1). It is further assumed that $\mathbf{H}(s) = \mathbf{G}(s)\mathbf{C}(s)$ is stable and proper. The primary objective is to relate BRG with some desired closed loop properties including the following,

Definition 4.4 The system $\mathbf{G}(s)$ is called to possess *integrity* [18, 30], if there exists a controller $\mathbf{K}(s)$ with integral action such that $\hat{\mathbf{K}}(s)$ stabilizes $\mathbf{G}(s)$ for all $\hat{\mathbf{K}}(\mathrm{s}) \in \mathcal{K}_I(s)$, where

$$\mathcal{K}_I(s) = \{\hat{\mathbf{K}}(s) = \epsilon\mathbf{K} \mid \epsilon \in \{0, 1\}\}$$

Definition 4.5 The system $\mathbf{G}(s)$ is called *block decentralized integral controllable* (block-DIC), if there exists a controller $\mathbf{K}(s)$ with integral action such that $\hat{\mathbf{K}}(s)$ stabilizes $\mathbf{G}(s)$ for all $\hat{\mathbf{K}}(s) \in \mathcal{K}_D(s)$, where

$$\mathcal{K}_D(s) = \{\hat{\mathbf{K}} = \mathrm{diag}(\epsilon_i \mathbf{I}_{m_i})\mathbf{K} \mid \epsilon_i \in [0, 1],\ \ i = 1, 2, \cdots, M\}$$

A system that possess integrity remains stable with integral action in every output channel, when any combination of loops is taken out of service. It is assumed that a controller that fails is immediately taken out of service, *i.e.* the corresponding entries in the block diagonal controller matrix are replaced by zero. The gain of the individual loops of a block-DIC system can be reduced independently of each other (or taken out of service) without introducing instability in the system. Note that Block-DIC is the block version of decentralized integral controllability (DIC) [18], known for fully decentralized controllers.

4.4.1 Stability

In this section, we consider the stability of the closed loop system operating under nominal conditions, with one or more loops open and in the presence of actuator failure. For fully decentralized control, it is well known that a system does not possess integrity, if one or more associated relative gains are negative. Grosdidier and Morari [50] have extended this result to block pairings.

Lemma 4.1 Let $\mathbf{H}(s) = \mathbf{GC}(s)$ be a rational proper system. With reference to Figure 4.1, assume that $k_1 = k_2 \cdots = k_M = k$. Then, $\mathbf{H}(s)$ is closed loop stable only if $\det(\mathbf{H}(0)) > 0$ [51].

Theorem 4.1 Let $\mathbf{H}(s)\mathbf{GC}(s)$ be a proper system. If $\det([\Lambda_B(0)]_{ii}) < 0$, for some i, $i = 1, 2 \cdots M$, then at least one of the following is true [50],

1. The closed loop system is unstable.

2. The closed loop system is unstable as i^{th} loop is removed.

3. The i^{th} loop considered in isolation with other loops is unstable.

Proof: Using (4.5) and Schur complement lemma, $\det([\mathbf{\Lambda}_B(0)]_{ii})$ can be expanded as,

$$\det([\mathbf{\Lambda}_B(0)]_{ii}) = \frac{\det(\mathbf{G}_{ii}(0))\det(\mathbf{G}^{ii}(0))}{\det(\mathbf{G}(0))} = \frac{\det(\mathbf{G}_{ii}(0)\mathbf{C}_{ii}(0))\det(\mathbf{G}^{ii}(0)\mathbf{C}^{ii}(0))}{\det(\mathbf{G}(0)\mathbf{C}(0))} \quad (4.15)$$

where $\mathbf{G}^{ii}(0)$ is the system $\mathbf{G}(0)$ with all the rows and columns corresponding to the i^{th} loop deleted. The second equality follows since $\det(\mathbf{C}(0)) = \det(\mathbf{C}_{ii}(0))\det(\mathbf{C}^{ii}(0))$. $\det([\mathbf{\Lambda}_B(0)]_{ii}) < 0$ implies that at least one of the terms in (4.15) is negative. Then, the conclusions can be drawn using Lemma 4.1. ■

If the individual loops are stable, then the stability of the closed loop system and the reduced system with one of the M loops removed is assessed using Theorem 4.1. It is generalized to the case when any combination of loops are open by the following corollary.

Corollary 4.1 Let p be a subset of integers chosen from the first M integers. Then, $\mathbf{G}_{pp}(0)$ is a submatrix consisting of blocks of $\mathbf{G}(0)$ indexed by p and $\mathbf{y}_p \in \mathbf{y}$. For a rational proper system $\mathbf{H}(s)$, if $\det([\mathbf{\Lambda}_B(\mathbf{G}_{pp}(0))]_{11}) < 0$, then at least one of the following is unstable:

1. the closed loop system or

2. the reduced system with the loops indexed by p removed ($\mathbf{y}_p$ left uncontrolled) or

3. the reduced system with only the loops indexed by p closed (only $\mathbf{y}_p$ controlled).

Though useful, when used alone, Theorem 4.1 can be inadequate in some cases. Consider the individual loops to be stable, but the closed loop system and the reduced system with i^{th} loop removed to be unstable. In this case, $\det([\mathbf{\Lambda}_B(0)]_{ii}) > 0$ despite the system not having integrity. This difficulty can be overcome by using Theorem 4.1 in conjunction with generalized Niederlinski index (NI).

Theorem 4.2 Let $\mathbf{H}(s)\mathbf{G}\mathbf{C}(s)$ be rational and proper. Assume that the individual loops are stable and have vanishing tracking error. Then the closed loop system is stable only if, [49]

$$\text{NI} = \frac{\det(\mathbf{G}(0))}{\prod_{i=1}^{M}\det(\mathbf{G}_{ii}(0))} > 0; \quad \det(\mathbf{G}_{ii}(0)) \neq 0 \;\; \forall i = 1, 2 \cdots M \qquad (4.16)$$

It follows from earlier discussion that a system has integrity only if $\det([\mathbf{\Lambda}_B(0)]_{ii}) > 0$ for all $i = 1, 2 \cdots M$ and NI > 0. Similar to loop failure sensitivity, an equally important issue is that of actuator failure sensitivity. A system is called j^{th} actuator failure sensitive (j-AFS), if the nominal system is stable but becomes unstable if the j^{th} actuator and j^{th} sensor are removed [51]. Then using Lemma 4.1, a system is $j-$AFS if $\det(\mathbf{G}^{jj}\mathbf{C}^{jj}) < 0$. Note that for SISO pairings, actuator failure sensitivity and loop failure sensitivity are equivalent. For block pairings, BRG can be used to assess the actuator failure sensitivity of the system. We assume that the variables of the system are reordered such that $1 \leq j \leq m_1$ or the j^{th} actuator lies in the first block of the partitioned system. Then, $[\mathbf{G}_{11}(0)]^{jj}$ is the loop gain with j^{th} sensor and j^{th} actuator removed.

Corollary 4.2 Let the rational proper system $\mathbf{H}(s)$ and its individual loops be nominally stable. Assume that $\det([\mathbf{\Lambda}_B(\mathbf{G}(0))]_{11}) > 0$. Then, if $\det([\mathbf{\Lambda}_B(\mathbf{G}^{jj}(0))]_{11}) < 0$ or $\mathrm{NI}(\mathbf{G}^{jj}(0)) < 0$, at least one of the following is $j-$AFS: (i) the closed loop system or (ii) the loop itself.

Proof: Similar to (4.15), $\det([\mathbf{\Lambda}_B(\mathbf{G}^{jj}(0))]_{11})$ can be expanded as,

$$\det([\mathbf{\Lambda}_B(\mathbf{G}^{jj}(0))]_{11}) = \frac{\det([\mathbf{G}_{11}(0)]^{jj}[\mathbf{C}_{11}(0)]^{jj})\det(\mathbf{G}'_{11}(0)\mathbf{C}'_{11}(0))}{\det(\mathbf{G}^{jj}(0)\mathbf{C}^{jj}(0))}$$

Since the nominal system and its individual loops are stable and $\det([\mathbf{\Lambda}_B(\mathbf{G}(0))]_{11}) > 0$, the reduced system with first loop removed is stable, *i.e.* $\det(\mathbf{G}'_{11}(0)\mathbf{C}'_{11}(0)) > 0$. Similarly,

$$\mathrm{NI}(\mathbf{G}^{jj}(0)) = \frac{\det(\mathbf{G}^{jj}(0))\mathbf{C}^{jj}(0))}{\det([\mathbf{G}_{11}(0)]^{jj}[\mathbf{C}_{11}(0)]^{jj})\prod_{i=2}^{M}\det(\mathbf{G}_{ii}(0))\mathbf{C}_{ii}(0))}$$

Since the individual loops are stable, $\prod_{i=2}^{M}\det(\mathbf{G}_{ii}(0))\mathbf{C}_{ii}(0)) > 0$. Now, the conclusions can be drawn using Lemma 4.1. ■

Remark 4.1 Chiu and Arkun [30] have shown that the system has integrity only if both BRG and NI, calculated for every possible combination of loops, are positive. For the same purposes, Häggblom [59] has also discussed a method based on the concept of Partial Relative Gains. Since the possible number of combinations of loops increases rapidly with system size, use of these methods (and Corollary 4.1) can be computationally expensive for systems beyond moderate dimensions. This issue is further discussed in the next chapter.

Remark 4.2 The similarity between Theorem 4.1 and IMC-filter stability criteria is noteworthy. Garcia and Morari [40] have shown that a sufficient condition for stability of a model inverse-based controller with a diagonal first order exponential filter is given by,

$$\mathrm{Re}\{\lambda_j(\mathbf{G}(0)\tilde{\mathbf{G}}(0)^{-1})\} > 0; \quad j = 1, 2 \cdots n \tag{4.17}$$

where $\tilde{\mathbf{G}}(s)$ is the nominal model of the system $\mathbf{G}$. Since, $\bar{\mathbf{G}}_{ii}(0) = [\mathbf{\Lambda}_B(0)]_{ii}^{-1}\mathbf{G}_{ii}(0)$, $[\mathbf{\Lambda}_B(0)]_{ii}^{-1}$ can be seen as multiplicative uncertainty in the i^{th} loop arising due to closure of all other loops. Based on (4.17), the i^{th} loop can be stabilized if $\mathrm{Re}\{\lambda_j(\mathbf{G}_{ii}(0)\bar{\mathbf{G}}_{ii}(0)^{-1})\} = \mathrm{Re}\{\lambda_j([\mathbf{\Lambda}_B(0)]_{ii})\} > 0$. Thus, a necessary (but not sufficient) condition for individual loop stability is $\det([\mathbf{\Lambda}_B(0)]_{ii}) > 0$, which is similar to Theorem 4.1. However, interpretation of BRG as multiplicative uncertainty is justified only if the effect of hidden feedback loops is small, which is not generally true.

4.4.2 Input Output Controllability

It is well known that right half plane (RHP) zeros close to the origin pose a limitation on the achievable output performance of the closed loop system. It is also possible that $\mathbf{G}_{ii}(s)$, considered in isolation, contains RHP zeros. The zeros of $\mathbf{G}_{ii}(s)$ can limit the achievable output performance, when the individual loops are designed independently. Skogestad and Hovd [65] have shown that the frequency dependent RGA can be used to detect the presence of RHP zeros (Theorem 1 in their paper). The applicability of their result is limited to the individual elements and $(n-1) \times (n-1)$ dimensional subsystems of $\mathbf{G}(s)$. The next proposition complements their result for subsystems having different dimensions.

Proposition 4.1 Consider a stable transfer function matrix $\mathbf{G}(s)$ and its partition in accordance to (4.2). Then $[\mathbf{\Lambda}_B(s)]_{11}$ would be an $m_1 \times m_1$ transfer function matrix. If there exists m_1, $2 \leq m_1 \leq n-2$, such that $\det([\mathbf{\Lambda}_B(j\infty)]_{11})$ is nonzero, finite and has a different sign from $\det([\mathbf{\Lambda}_B(0)]_{11})$, then at least one of the following is true,

(a) The subsystem $\mathbf{G}_{11}$ has a RHP zero.

(b) The subsystem $\mathbf{G}_{22}$ has a RHP zero.

Proof: For a given partitioning of the system, $2 \leq m_1 \leq n-2$, consider that $\lim_{s \to j\infty} \det([\mathbf{\Lambda}_B(s)]_{11})$ is nonzero and finite. If the signs of $\det([\mathbf{\Lambda}_B(0)]_{11})$ and $\lim_{s \to j\infty} \det([\mathbf{\Lambda}_B(s)]_{11})$ are different, then there exists a frequency ω_o, $\omega_o > 0$, such that $\det([\mathbf{\Lambda}_B(j\omega_o)]_{11}) = 0$.

The equality, $\det([\mathbf{\Lambda}_B(s)]_{11}) = 0$, is satisfied, iff one or both of $\det(\mathbf{G}_{11}(j\omega_o))$ and $\det(\bar{\mathbf{G}}_{11}^{-1}(j\omega_o))$ are zero. Now, $\det(\mathbf{G}_{11}(j\omega_o))$ being zero implies the presence of a RHP zero in $\mathbf{G}_{11}(s)$ at that frequency.

If $\det(\bar{\mathbf{G}}_{11}^{-1}(j\omega_o)) = 0$, then $\bar{\mathbf{G}}_{11}^{-1}(s)$ contains an RHP zero and $\bar{\mathbf{G}}_{11}(s)$ contains an RHP pole at that frequency. Due to stability assumptions, an RHP pole in $\bar{\mathbf{G}}_{11}(s)$ at $s = j\omega_o$ can arise only due to an RHP zero in $\mathbf{G}_{22}(s)$ at $s = j\omega_o$. ■

This result is equally valid, if $m_1 = 1$ or $n - 1$. Then, $\mathbf{G}_{11}(s)$ or $\mathbf{G}_{22}(s)$ are single elements of $\mathbf{G}(s)$. In this case, if the condition imposed by Proposition 4.1 is satisfied, one or both of $\mathbf{G}_{11}(s)$ and $\mathbf{G}_{22}(s)$ will contain RHP zero. The BRG is input scaling independent (see Property 4.3). Thus, if an input channel of $\mathbf{G}(s)$ contains an RHP zero, the signs of $\det([\mathbf{\Lambda}_B(j\infty)]_{11})$ and $\det([\mathbf{\Lambda}_B(0)]_{11})$ will remain unchanged. The change of sign of $\det([\mathbf{\Lambda}_B(s)]_{11})$ is only a sufficient, but not a necessary condition for the presence of RHP zeros in the subsystems of $\mathbf{G}(s)$.

Corollary 4.3 Consider that $\mathbf{G}_{22}(s)$ contains a RHP zero. If all loops but $(\mathbf{y_1}(s), \mathbf{u_1}(s))$ are closed, then the open loop subsystem $(\mathbf{y_1}(s), \mathbf{u_1}(s))$ or $\bar{\mathbf{G}}_{11}(s)$ contains a RHP pole.

If a RHP pole appears in the $(\mathbf{y_1}(s), \mathbf{u_1}(s))$ loop due to closure of all other loops, any small disturbance in that open loop can destabilize the system. In practice, however, the gain of the loop would remain finite due to presence of physical constraints.

Proposition 4.1 excludes the case in which any of the subsystems contain a zero at origin, $(s = 0)$. Should a subsystem contain a zero at the origin, it would be extremely difficult to control the system. The relation between zeros at origin and the steady state BRG is established in the next corollary.

Corollary 4.4 If there exists m_1, $\{m_1 = 1, \cdots, n-1\}$, such that $\det([\mathbf{\Lambda}_B(0)]_{11}) = 0$, then one or both of the subsystems, $\mathbf{G}_{11}(s)$ and $\mathbf{G}_{22}(s)$ contain a zero or a zero at the origin.

Either of these conditions is highly undesirable because it makes the system uncontrollable. The system may also contain zeros close to the origin in the open LHP. The presence of such poorly damped zeros also affect the system's controllability. In such cases, the gain of the individual loops increases considerably with closure of all other loops.

The gain of a multivariate system depends on the input direction. Let the gain of $(\mathbf{y_1}(s), \mathbf{u_1}(s))$ be $\|\mathbf{G}_{11}(0)\mathbf{v}\|_2$, $\|\mathbf{v}\|_2 = 1$. Similarly, let the apparent gain of this loop, when all other loops are closed be $\|\bar{\mathbf{G}}_{11}(0)\mathbf{w}\|_2$, $\|\mathbf{w}\|_2 = 1$.

Proposition 4.2 The worst case gain mismatch between $\mathbf{G}_{11}(0)$ and $\bar{\mathbf{G}}_{11}(0)$ is bounded as follows,

$$\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{11}) \leq \max_{\substack{\|\mathbf{v}\|_2=1 \\ \|\mathbf{w}\|_2=1}} \frac{\|\mathbf{G}_{11}(0)\mathbf{v}\|_2}{\|\bar{\mathbf{G}}_{11}(0)\mathbf{w}\|_2} \tag{4.18}$$

$$\frac{1}{\underline{\sigma}([\mathbf{\Lambda}_B(0)]_{11})} \leq \max_{\substack{\|\mathbf{v}\|_2=1 \\ \|\mathbf{w}\|_2=1}} \frac{\|\bar{\mathbf{G}}_{11}(0)\mathbf{w}\|_2}{\|\mathbf{G}_{11}(0)\mathbf{v}\|_2} \tag{4.19}$$

Proof: For (4.18),

$$\begin{aligned}\max_{\substack{\|\mathbf{v}\|_2=1 \\ \|\mathbf{w}\|_2=1}} \frac{\|\mathbf{G}_{11}(0)\mathbf{v}\|_2}{\|\bar{\mathbf{G}}_{11}(0)\mathbf{w}\|_2} &= \frac{\bar{\sigma}(\mathbf{G}_{11}(0))}{\underline{\sigma}(\bar{\mathbf{G}}_{11}(0))} = \bar{\sigma}(\mathbf{G}_{11}(0))\bar{\sigma}(\bar{\mathbf{G}}_{11}^{-1}(0)) \\ &\geq \bar{\sigma}([\mathbf{\Lambda}_B(0)]_{11})\end{aligned}$$

For (4.19),

$$\begin{aligned}\max_{\substack{\|\mathbf{v}\|_2=1 \\ \|\mathbf{w}\|_2=1}} \frac{\|\bar{\mathbf{G}}_{11}(0)\mathbf{w}\|_2}{\|\mathbf{G}_{11}(0)\mathbf{v}\|_2} &= \frac{\bar{\sigma}(\bar{\mathbf{G}}_{11}(0))}{\underline{\sigma}(\mathbf{G}_{11}(0))} = \bar{\sigma}(\bar{\mathbf{G}}_{11}(0))\bar{\sigma}(\mathbf{G}_{11}^{-1}(0)) \geq \bar{\sigma}([\mathbf{\Lambda}_B(0)]_{11}^{-1}) \\ &\geq \frac{1}{\underline{\sigma}([\mathbf{\Lambda}_B(0)]_{11})}\end{aligned}$$

■

Proposition 4.2 suggests that if at least one of the following conditions, $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{11}) \gg 1$ and $\underline{\sigma}([\mathbf{\Lambda}_B(0)]_{11}) \ll 1$, is satisfied, then the gain of $\mathbf{y}_1(s)-\mathbf{u}_1(s)$ loop changes considerably due to closure of all other loops. If $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{11}) \approx 1$ and $\underline{\sigma}([\mathbf{\Lambda}_B(0)]_{11}) \approx 1$, the change in gain may still be large, as (4.18) and (4.19) are lower bounds on the worst case gain mismatch with one of the loops open. This affirms our earlier assertion that if the BRG is far from the Identity matrix, the system has large interactions, but the converse is not true. This is further discussed in §4.4.4.

4.4.3 Block Diagonal Dominance

When the system is block diagonal or triangular, the individual controllers can be tuned independently of each other (Property 4.5); however, most real systems do not lie in this class. Independent tuning of individual controllers to give stable closed loop response is still possible, if the effect of $\mathbf{u}_i$ on $\mathbf{y}_i$ is large compared to the effect of $\mathbf{u}_j$, $(i \neq j)$. The concept of block diagonal dominance can be used to assess this property of the partitioned system.

Definition 4.6 A matrix $\mathbf{Z}$ is *generalized row block diagonal dominant* for a given partitioning if there exists $\mathbf{x} \in \mathbb{R}^M$, $\mathbf{x} > 0$ such that [78],

$$\| \mathbf{Z}_{ii}^{-1} \|^{-1} \mathbf{x}_i > \sum_{j=1,j\neq i}^{M} \| \mathbf{Z}_{ij} \| \mathbf{x}_j; \quad i = 1, 2, \cdots, M$$

Generalized column block dominance is defined similarly. If $\mathbf{x}$ can be chosen as $\mathbf{1}_n^T$, then $\mathbf{Z}$ is called row (column) block dominant. If $\mathbf{Z}$ is generalized block diagonal dominant (GBDD), there exists a scaling matrix of the form $\mathbf{X} = \mathrm{diag}(x_i \mathbf{I}_{m_i})$, $i = 1, 2 \cdots M$ such that $\mathbf{XZX}^{-1}$ is block diagonal dominant [78].

Limbeer [78] has shown that if $(\mathbf{I}_n + \mathbf{GK}(s))$ is GBDD for all s, then the stability of individual loops implies the stability of the closed loop system. When the controller contains integral action, $\mathbf{I}_n + (1/s)\mathbf{H}(s) \approx (1/s)\mathbf{H}(s)$ at low frequencies [65]. At these frequencies, the diagonal dominance of $(\mathbf{I}_n + (1/s)\mathbf{H}(s))$ can be assessed from diagonal dominance of $\mathbf{H}(s)$. In addition, if a system is GBDD at steady state, it is also block-DIC, as shown next. Here, p is an ordered subset of integers chosen within the set $\{1, 2, \cdots M\}$ and $\tilde{\mathcal{I}}$ is the ensemble of all possible p's. Then, $\mathbf{G}_{pp}(0)$ is a submatrix consisting of blocks of $\mathbf{G}(0)$ indexed by p.

Lemma 4.2 Let H(s) be a proper system and the matrix $\mathbf{D} \in \mathbb{R}^{n \times n}$ be defined as $\mathbf{D} = \mathrm{diag}(d_i \mathbf{I}_{m_i})$, $d_i > 0$. Then, G(s) is block-DIC, iff there exists a block diagonal matrix $\mathbf{C}(\mathbf{0})$ such that

$$\mathrm{Re}\{\lambda_j([\mathbf{DH}(0)]_{pp})\} > 0 \quad \forall j, \forall p \in \tilde{\mathcal{I}}$$

Proof: Campo and Morari [18] have shown that a similar condition is necessary and sufficient for a system to be DIC. This lemma can be shown to be true following their proof. ■

Proposition 4.3 Let $\mathbf{H}(s)$ be a proper stable system. If $\mathbf{H}(0)$ is block diagonally dominant, then G(s) is block-DIC.

Proof: With reference to Figure 4.1, let $\mathbf{G}_{ii}(s)\mathbf{K}_{ii}(s) = (k_i/s \cdot \mathbf{I}_{m_i})\mathbf{H}_{ii}(s)$. The i^{th} loop will be stable iff any of the characteristic loci of $(k_i/s)\mathbf{H}_{ii}(s)$ does not encircle the point $(-1/k_i, 0)$, as s traverses the Nyquist D-contour. Since $\mathbf{H}_{ii}(s)$ is stable by assumption, such an encirclement can occur only due to the pole at the origin. Grosdidier *et al.* [51] have shown that as $k_i \to 0$, the j^{th} characteristic loci does not cross the negative real axis if $\mathrm{Re}\{\lambda_j(\mathbf{H}_{ii}(0))\} > 0$; $j = 1, 2 \cdots m_i$.

For a block diagonal dominant system, the total number of encirclements are same as the sum of encirclements by individual loops [78]. Then , $\mathrm{Re}\{\lambda_j(\mathbf{H}(0))\} > 0; \quad j = 1, 2 \cdots n$, if

$$\mathrm{Re}\{\lambda_j(\mathbf{H}_{ii}(0)\} > 0; \quad \forall j = 1, 2 \cdots m_i, \ \forall i = 1, 2 \cdots M \tag{4.20}$$

Note that $\mathbf{H}_{pp}(0)$ is block diagonal dominant due to block diagonal dominance of $\mathbf{H}(0)$ for all $p \in \tilde{\mathcal{I}}$. Then $\mathrm{Re}\{\lambda_j([\mathbf{DH}(0)]_{pp})\} > 0$ for all $p \in \tilde{\mathcal{I}}$ and the system is block-DIC, if (4.20) is satisfied. For a block diagonal dominant system, (4.20) can always be satisfied by choosing $\mathbf{C}_{ii}(0) = \mathbf{G}_{ii}^{-1}(0)$, where the invertibility of $\mathbf{G}_{ii}^{-1}(0)$ is guaranteed by block diagonal dominance. ■

In order to verify the generalized block diagonal dominance of $\mathbf{H}_{ii}(0)$, knowledge of the compensator matrix $\mathbf{C}_{ii}(0)$ is required, which can be limiting for practical purposes. We show that whether $\mathbf{H}_{ii}(0)$ is GBDD, can be assessed using BRG, which is independent of the compensator matrix. Though the following results are valid for any matrix norms, we use the induced 2-norm due to their frequent use in the process control literature.

Lemma 4.3 Let $\mathbf{Z}$ be GBDD. Then [78],

$$\bar{\sigma}(\left[\mathbf{Z}^{-1}\right]_{ii}) \geq \bar{\sigma}(\left[\mathbf{Z}^{-1}\right]_{ji}) \quad \forall i \neq j$$

Proposition 4.4 The system $\mathbf{H}(s)$ is GBDD only if

$$\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) > 0.5 \ \ \forall i = 1, 2 \cdots M \tag{4.21}$$

Proof: This proposition is proved using the following logical identity: If $A \Rightarrow B$, then not $B \Rightarrow$ not A. If $\mathbf{H}(0)$ is a block diagonal dominant matrix

$$\underline{\sigma}(\mathbf{H}_{ii}(0)) \ > \ \sum_{j=1, j\neq i}^{M} \bar{\sigma}(\mathbf{H}_{ij}(0))$$

Then, using Lemma 4.3,

$$\begin{aligned}
\underline{\sigma}(\mathbf{H}_{ii}(0))\bar{\sigma}(\left[\mathbf{H}^{-1}(0)\right]_{ii}) \ &> \ \sum_{j=1, j\neq i}^{M} \bar{\sigma}(\mathbf{H}_{ij}(0))\bar{\sigma}(\left[\mathbf{H}^{-1}(0)\right]_{ji}) \\
\bar{\sigma}(\mathbf{H}_{ii}(0)\left[\mathbf{H}^{-1}(0)\right]_{ii}) \ &> \ \sum_{j=1, j\neq i}^{M} \bar{\sigma}(\mathbf{H}_{ij}(0)\left[\mathbf{H}^{-1}(0)\right]_{ji}) \\
\bar{\sigma}(\mathbf{G}_{ii}(0)\left[\mathbf{G}^{-1}(0)\right]_{ii}) \ &> \ \sum_{j=1, j\neq i}^{M} \bar{\sigma}(\mathbf{G}_{ij}(0)\left[\mathbf{G}^{-1}(0)\right]_{ji})
\end{aligned} \tag{4.22}$$

Consider that multiplication of $\mathbf{G}(0)$ with $\mathbf{G}^{-1}(0)$,

$$\sum_{j=1}^{M} \mathbf{G}_{ij}(0)\left[\mathbf{G}^{-1}(0)\right]_{ji} = \mathbf{I}_{m_i}$$

$$\bar{\sigma}(\mathbf{G}_{ii}(0)\left[\mathbf{G}^{-1}(0)\right]_{ii}) + \sum_{j=1,j\neq i}^{M} \bar{\sigma}(\mathbf{G}_{ij}(0)\left[\mathbf{G}^{-1}(0)\right]_{ji}) \geq 1$$

Then, using the definition of BRG (4.5) and (4.22), $\mathbf{H}(0)$ is block diagonal dominant only if $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) > 0.5 \;\; \forall i = 1, 2 \cdots M$. Since BRG is independent of scaling of the form $\mathbf{X} = \text{diag}(x_i \mathbf{I}_{m_i})$ (see property 4.3), (4.22) is necessary for the system to be GBDD. ■

Ohta *et al.* [88] have pointed out that in many cases, GBDD can be a very conservative test for block diagonal dominance and have instead suggested the use of quasi-block diagonal dominance (QBDD). They have shown that if $(\mathbf{I}_n + \mathbf{GK}(s))$ is QBDD for all s, then the stability of individual loops implies the stability of the closed loop system. In the following discussion, QBDD is defined formally and it is shown that the condition $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) > 0.5$ for all $i = 1, 2 \cdots M$ is necessary for a system to be QBDD.

Definition 4.7 A matrix $\mathbf{Z}$ is *quasi-block diagonal dominant* for a given partitioning if there exists $\mathbf{x} \in \mathbb{R}^M$ such that,

$$\mathbf{x}_i > \sum_{j=1,j\neq i}^{M} \parallel \mathbf{Z}_{ij}\mathbf{Z}_{ii}^{-1} \parallel \mathbf{x}_j; \;\; i = 1, 2, \cdots, M; \;\; \mathbf{Z}_{ii} \neq \mathbf{0}$$

Corollary 4.5 The system $\mathbf{H}(s)$ is QBDD only if $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) > 0.5$ for all $i = 1, 2 \cdots M$.

Proof: At low frequencies, $(\mathbf{I}_n + (1/s)\mathbf{H}(s)) \approx (1/s)\mathbf{H}(s)$. Then, $\mathbf{H}_{ij}(0)\mathbf{H}_{ii}^{-1}(0) = \mathbf{G}_{ij}(0)\mathbf{G}_{ii}^{-1}(0)$. When the compensator matrix is chosen as $\mathbf{C}_{ii} = \mathbf{G}_{ii}^{-1}(0)$, $i = 1, 2 \cdots M$, GBDD and QBDD are equivalent (by definition). Then, using Proposition 4.4, $\mathbf{H}(0)$ is QBDD only if $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) > 0.5$, for all $i = 1, 2 \cdots M$. ■

Let $\mathbf{E}(s) = (\mathbf{H}(s)\mathbf{H}_{bd}(s)^{-1} - \mathbf{I}_n) = (\mathbf{G}(s)\mathbf{G}_{bd}(s)^{-1} - \mathbf{I}_n)$, where $\mathbf{H}_{bd}(s)$ and $\mathbf{G}_{bd}(s)$ are matrices containing the block diagonal elements of $\mathbf{H}(s)$ and $\mathbf{G}(s)$ respectively (see Figure 4.2). Ohta *et al.* [88] have shown that if the $\mathbf{H}(s)$ is QBDD, there exists a norm such that $\parallel \mathbf{XE}(s)\mathbf{X}^{-1} \parallel < 1$, where $\mathbf{X}$ is the scaling matrix, defined as before. Let $\mathcal{X}$ be the set defined as

$$\mathcal{X} = \{\mathbf{X} \in \mathbb{R}^{n \times n} \mid \mathbf{X}_{ii} = \text{diag}(x_i \cdot \mathbf{I}_{m_i})\} \;\; i = 1, 2 \cdots M$$

Then, with the choice of induced 2–norm, the following relation holds,

$$\mu_{\mathbf{\Delta}}(\mathbf{E}(s)) \leq \inf_{\mathbf{X}\in\mathcal{X}} \bar{\sigma}(\mathbf{X}\mathbf{E}(s)\mathbf{X}^{-1}) \quad \bar{\sigma}(\mathbf{\Delta}) \leq 1$$

where $\mu_{\mathbf{\Delta}}(.)$ is the structured singular value. The structure of $\mathbf{\Delta}$ can be chosen to be the same as that of $\mathbf{G}_{bd}(s)$, since every $\mathbf{X} \in \mathcal{X}$ commutes with $\mathbf{\Delta}$, *i.e.* $\mathbf{X\Delta} = \mathbf{\Delta X}$. Then (with the choice of induced 2–norm), the system $\mathbf{H}(s)$ is QBDD or GBDD only if $\mu_{\mathbf{\Delta}}(\mathbf{E}(0)) < 1$. Note that this condition is also sufficient for block diagonal dominance of the system at steady state.

In a related context, Grosdidier and Morari [49] defined $\mu_{\mathbf{\Delta}}(\mathbf{E}(s))$ as the μ-IM to assess the *closeness* of $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$. They have shown that if $\mu_{\mathbf{\Delta}}(\mathbf{E}(0)) > 1$, a block diagonal controller with integral action cannot be designed for the given system. For fully decentralized control, Braatz [11] has shown that a system is DIC, if $\mu_{\mathbf{\Delta}}(\mathbf{E}(0)) < 1$. This result can be easily extended to the block decentralized controllers. Whereas a pairing alternative that satisfies the μ interaction condition is guaranteed to have some attractive properties, the computational load for the calculation of μ is large [15, 39]. Noting that for all $\mathbf{X} \in \mathcal{X}$, $\mathbf{X}_{ii}[\mathbf{\Lambda}_B(0)]_{ii}\mathbf{X}_{ii}^{-1} = [\mathbf{\Lambda}_B(0)]_{ii}$ (see Property 4.3), the following useful result is obtained:

Corollary 4.6 For a proper system $\mathbf{G}(s)$, $\mu_{\mathbf{\Delta}}(\mathbf{E}(s)) < 1$ only if $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) > 0.5$ for all $i = 1, 2 \cdots M$.

For fully decentralized control, the necessary condition $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) > 0.5$ reduces to $\lambda_{ii} > 0.5$ for all $i = 1, 2 \cdots M$. Grosdidier and Morari [49] have shown this result to be true for 2 × 2 systems and Corollary 4.6 can be seen as generalization of this result to systems with higher dimensions and block decentralized controllers. Corollary 4.6 can be used for pre-screening the alternatives for pairings, reducing the computational load significantly.

4.4.4 Closed Loop Interactions

In Figure 4.2, if $\mathbf{G}(s) = \mathbf{G}_{bd}(s)$, the system is trivially *non-interacting*. In this section, such a system is referred to as an *ideal* system. When the controller contains integral action, at low frequencies, the sensitivity functions of the actual and the ideal systems are related as [102],

$$\begin{aligned} \mathbf{S}(s) &\approx \mathbf{S}_{bd}(s)\mathbf{\Gamma}(s) \\ \mathbf{S}(s) &= (\mathbf{I}_n + \mathbf{G}(s)\mathbf{K}(s))^{-1} \\ \mathbf{S}_{bd}(s) &= (\mathbf{I}_n + \mathbf{G}_{bd}(s)\mathbf{K}(s))^{-1} \end{aligned}$$

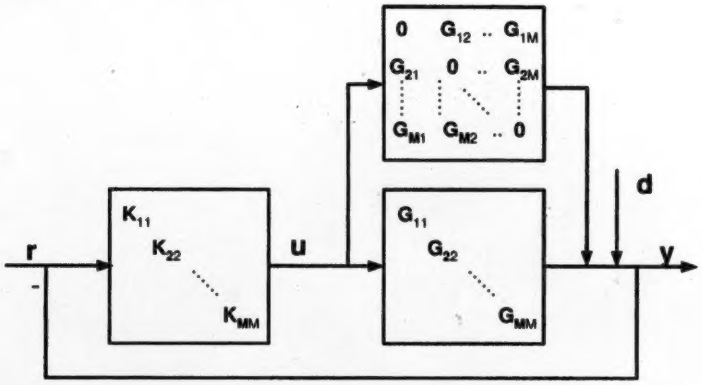

Figure 4.2: Decomposition of system into block diagonal and off-block diagonal elements

where $\mathbf{\Gamma}(s) = \mathbf{G}_{bd}(s)\mathbf{G}^{-1}(s)$ is the Performance Relative Gain Array (PRGA) [65]. $\mathbf{\Gamma}(s)$ can be interpreted as a filter that amplifies and rotates the exogenous signals. This action prevents the actual system from behaving as the ideal system. Let $\mathbf{\Gamma}(s)$ be expressed through its singular value decomposition as, $\mathbf{\Gamma}(s) = \mathbf{U}(s)\mathbf{\Sigma}(s)\mathbf{V}^T(s)$. Then,

$$\mathbf{\Gamma}(s)\mathbf{v}_i(s) = \sigma_i(s)\mathbf{u}_i(s), \quad \forall i = 1, 2 \cdots n$$

where $\sigma_i(s)$ is the i^{th} singular value and $\mathbf{u}_i(s)$ and $\mathbf{v}_i(s)$ are the corresponding left and right singular vectors, calculated at a particular frequency. Grosdidier [48] has argued that the exogenous signals oriented in the direction of singular vectors associated with $\bar{\sigma}(\mathbf{\Gamma}(s))$ most adversely affect the closed loop performance. Then, for minimization of worst case performance loss, we may require that $\bar{\sigma}(\mathbf{\Gamma}(s))$ be minimum in the desired frequency range. Similarly, a necessary condition for interactions to be minimum is that $\sigma_i(\mathbf{\Gamma}(s)) \approx 1$, for all $i = 1, 2, \cdots n$ in the desired frequency range. If this happens, then at every frequency, $\mathbf{\Gamma}(s)$ is close to a unitary matrix; however,

$$\max_i\{\bar{\sigma}([\mathbf{\Lambda}_B(s)]_{ii})\} \leq \bar{\sigma}(\mathbf{\Gamma}(s)) \quad i = 1, 2, \cdots M \tag{4.23}$$

Therefore, if $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) \gg 1$, for all $i = 1, 2, \cdots M$, then $\bar{\sigma}(\mathbf{\Gamma}(0)) \gg 1$. When $[\mathbf{\Lambda}_B(0)]_{ii}$ $= \mathbf{I}$, then $\sigma_j([\mathbf{\Lambda}_B(0)]_{ii}) = 1$, for all $i = 1, 2, \cdots M, j = 1, 2, \cdots m_i$. Then, (4.23) suggests that $\bar{\sigma}(\mathbf{\Gamma}(0))$ can still be large, despite the BRG being precisely the Identity matrix. Based on these observations and Proposition 4.2, we conclude that the system has large interactions, if $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) \gg 1$ and $\underline{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) \ll 1$ or in other words, *BRG is very different from Identity*, but the converse is not true. Thus, use of the PRGA is necessary for drawing any conclusions regarding closed loop interactions. In some cases, this measure can be conservative, as it does not take the directional information of $\mathbf{S}_{bd}$ into account.

Remark 4.3 The requirement that $\sigma_i(\mathbf{\Gamma}(s)) \approx 1$ for low interactions is equivalent to minimization of $\sum_i |\sigma_i(\mathbf{\Gamma}(s)) - 1|$. In most of cases, it is seen that $\sum_i |\sigma_i(\mathbf{\Gamma}(s)) - 1| \approx \bar{\sigma}(\mathbf{\Gamma}(s))$. Let the norm of exogenous signals be bounded from above by 1. Then for the feedback to be effective, we require that $\bar{\sigma}(\mathbf{S}(s)) < 1$ in the desired frequency range, which is lower bounded by $\underline{\sigma}(\mathbf{S}_{bd}(s))\bar{\sigma}(\mathbf{\Gamma}(s))$ at low frequencies. Then, $\bar{\sigma}(\mathbf{S}(s)) < 1$ only if $\underline{\sigma}(\mathbf{S}_{bd}(s))\bar{\sigma}(\mathbf{\Gamma}(s)) < 1$ or $\bar{\sigma}(\mathbf{I} + \mathbf{G}_{bd}(s)\mathbf{K}(s)) > \bar{\sigma}(\mathbf{\Gamma}(s))$. This inequality can be easily satisfied by choosing a controller with low gain if $\bar{\sigma}(\mathbf{\Gamma}(s))$ is small. Large controller gains may present operational difficulties in presence of input constraints.

4.5 Alternate Pairing Rules

In earlier sections, it was shown that useful information regarding many closed loop properties can be extracted using the BRG. In this section, we summarize those results in the form of pairing rules.

Pairing Rule 1 Avoid pairing on variables, with $\det([\mathbf{\Lambda}_B(0)]_{ii}) \leq 0$ for some i or NI < 0, otherwise the system does not have integrity (See Theorems 4.1,4.2 and Corollary 4.4).

Pairing Rule 2 Prefer pairing on variables for which $\mu_{\mathbf{\Delta}}(\mathbf{E}(0)) < 1$. Alternatives satisfying this condition are decoupled at low frequencies and a block decentralized controller with integral action can be designed easily (See §4.4.3). The associated computational load can be reduced by pre-screening alternatives such that $\bar{\sigma}([\mathbf{\Lambda}_B(0)])_{ii} > 0.5$ for all i (See Proposition 4.4 and Corollary 4.6).

Pairing Rule 3 Prefer pairing on variables for which $J(0) = \sum_i |\sigma_i(\mathbf{\Gamma}(0)) - 1|$ is small. If $J(0)$ is small, then the system is *weakly* interacting and vice versa, at least at steady state (See §4.4.4).

These rules are based on gain information only and may suggest inferior pairings for systems containing large time delays. In such cases, if a reliable dynamic model is available, then ensuring that $J(s) = \sum_i |\sigma_i(\mathbf{\Gamma}(s)) - 1|$ is small up to the crossover frequency is helpful. In addition,

Pairing Rule 4 Avoid pairing on variables with different signs of $\det([\mathbf{\Lambda}_B(0)]_{ii})$ and $\det([\mathbf{\Lambda}_B(j\infty)]_{ii})$. If the signs are different, then the i^{th} loop or the remaining subsystem contains an RHP zero (See Proposition 4.1).

Remark 4.4 Since BRG and PRGA are output scaling dependent, so are their singular values. Therefore, prior to pairing selection, specification of a suitable scaling of the system matrix is necessary to avoid ambiguity. Some possible approaches are to normalize the system matrix such that $\|y_i\|_2 \leq 1$ or $|y_i| \leq 1$.

Remark 4.5 These pairing rules equally hold for fully decentralized control structures. For many problems, $\sum_i |\sigma_i(\mathbf{\Gamma}(0)) - 1|$ is small, if the diagonal elements of RGA elements are close to 1. Thus, Bristol's rule of pairing on RGA elements close to 1 is implicit here, but, in general, it is neither necessary nor sufficient for the system to be weakly interacting.

Remark 4.6 Often, $\sum_i |\sigma_i(\mathbf{\Gamma}(0)) - 1|$ approaches zero monotonically as the controller structure approaches the *fully centralized* case. In such cases, a balance should be made between the closed loop performance and the controller complexity. If a more complex controller structure shows no significant performance improvement, then the simpler structure (closer to the *fully decentralized* case) should be preferred.

4.5.1 Numerical Examples

Example 4.3 Consider the 4×4 ALSTOM gasifier system [32]. The gasifier is described by three linearized state space models of 25^{th} order at 100%, 50% and 0% load conditions. Prior to pairing selection, the system is scaled. The scaling procedure and the scaled gain matrices are given in the Appendix 4.A.

Various alternatives are screened at different load conditions. The analysis suggests that $((1-2-4, 1-3-4), (3-2))$[3] is the only alternative, which satisfies Rules 1 and 2 at all load conditions. Since $\mu_{\mathbf{\Delta}}(\mathbf{E}(0)) < 1$ for this alternative, the blocks are decoupled at low frequencies and a controller with integral action can be designed easily.

This system has also been analyzed by Chin and Munro [29] at 100% load conditions, where they have suggested the use of $((1-3-4, 2-3-4), (2-1))$. This alternative satisfies Rules 1 and 2 at 100% load conditions, but the relative gain of the pairing $(2-1)$ is negative at 0% load conditions. This shows that this alternative will lose integrity under varying operating conditions. Though Chin and Munro [29] have scaled the system differently, it has no effect on the conclusions, since $\det([\mathbf{\Lambda}_B]_{ii})$ is independent of scaling.

Example 4.4 In most of the case studies, we have found steady state analysis to be sufficient, but in some cases it may suggest inferior pairings, as shown here. Alatiqi and

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[3] $((1-2-4, 1-3-4), (3-2))$ represents $((y_1 - y_2 - y_4, u_1 - u_3 - u_4), (y_3, u_2))$ variable pairing.

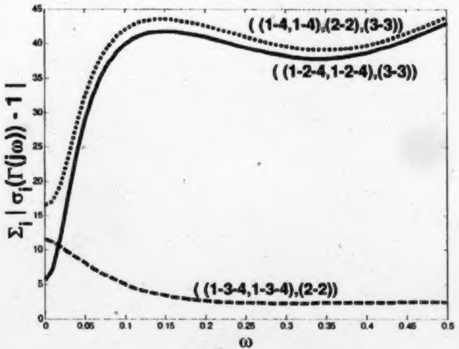

Figure 4.3: $\sum_i |\sigma_i(\mathbf{\Gamma}(j\omega)) - 1|$ for Column/Stripper Distillation system

Luyben [1] considered the following column/stripper distillation system,

$$\mathbf{G}(s) = \begin{pmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-6.36e^{-1.2s}}{(31.6s+1)(20s+1)} & \frac{-0.25e^{-1.4s}}{(21s+1)} & \frac{-0.49e^{-6s}}{(22s+1)^2} \\ \frac{-4.17e^{-5s}}{(45s+1)} & \frac{6.93e^{-1.02s}}{(44.6s+1)} & \frac{-0.05e^{-6s}}{(34.5s+1)^2} & \frac{1.53e^{-3.8s}}{(48s+1)} \\ \frac{1.73e^{-18s}}{(13s+1)^2} & \frac{5.11e^{-12s}}{(13.3s+1)^2} & \frac{4.61e^{-1.01s}}{(18.5s+1)} & \frac{-5.49e^{-1.5s}}{(15s+1)} \\ \frac{-11.2e^{-2.6s}}{(43s+1)(6.5s+1)} & \frac{14(10s+1)e^{-0.02s}}{(45s+1)(17.4s^2+3s+1)} & \frac{0.1e^{-0.05s}}{(31.6s+1)(5s+1)} & \frac{4.49e^{-0.6s}}{(48s+1)(6.3s+1)} \end{pmatrix}$$

The alternatives are screened using the suggested pairing rules and all the alternatives satisfying Rules 1 and 2 are summarized in Table 4.2. Based on steady state analysis, it might seem that $((1-2-4, 1-2-4), (3-3))$ is the best structure, but its performance deteriorates considerably at higher frequencies. Figure 4.3 shows $\sum_i |\sigma_i(\mathbf{\Gamma}(s)) - 1|$ as a function of frequency for different structures. At moderate frequencies, $((1-3-4, 1-3-4), (2-2))$ gives improved performance as compared to other alternatives and thus its use is recommended. It should be noted that no viable alternative exists for $2 \times 2/2 \times 2$ decomposition of the system. Block decentralized structures close to the fully centralized case need not always be better than simpler structures as previously pointed out by Manousiouthakis *et al.* [83].

4.6 Note on Integrating Systems

The RGA, as originally defined by Bristol [17], is applicable to only open loop stable processes. Arkun and Downs [4] have shown that it is still possible to use the RGA, when

Pairing	$\min_i(\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}))$	$\mu_\Delta(\mathbf{E}(0))$	$\sum_i \lvert\sigma_i(\mathbf{\Gamma}(0)) - 1\rvert$
(1-4,1-4),(2,2),(3,3)	1.19	0.96	16.59
(1-2-4,1-2-4),(3-3)	1.19	0.53	5.65
(1-3-4,1-3-4),(2-2)	0.92	0.94	11.52

Table 4.2: Alternatives for decentralized control of Column/Stripper Distillation system

the system contains integrating elements in one or more input or output channels. In such cases, the RGA is calculated by replacing the elements containing the integrators by their derivatives. Here, we investigate the applicability of this approach for the BRG.

Case I: Consider the case, when one or more input channels (columns of $\mathbf{G}(s)$) contain integrator. Then, the system matrix can be partitioned into non-integrating ($\mathbf{G}_{NI}(s)$) and integrating ($\mathbf{G}_I(s)$) blocks as,

$$\mathbf{G}(s) = \begin{bmatrix} \mathbf{G}_{NI}(s) & \frac{1}{s}\mathbf{G}_I(s) \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{NI}(s) & \mathbf{G}_I(s) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{s}\mathbf{I} \end{bmatrix} \tag{4.24}$$

If the second block in (4.24) is treated as a scaling matrix, then $[\mathbf{\Lambda}_B^s(0)]_{ii} = [\mathbf{\Lambda}_B(0)]_{ii}$ (Property 4.3(i)). In this case, it would be possible to select block pairings such that the individual blocks contain both integrating and non-integrating elements.

Case II: Now, consider the case, when one or more output channels (rows of $\mathbf{G}(s)$) contain integrators. Partitioning the system matrix as before,

$$\mathbf{G}(s) = \begin{bmatrix} \mathbf{G}_{NI}(s) \\ \frac{1}{s}\mathbf{G}_I(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \frac{1}{s}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{G}_{NI}(s) \\ \mathbf{G}_I(s) \end{bmatrix} \tag{4.25}$$

Here, any meaningful results can be obtained only if all the outputs containing integrators are paired together (Property 4.3(iii)) or if paired separately, only SISO pairing is used for them (Property 4.3(ii)). No block pairing should contain both non-integrating and integrating elements.

4.7 Chapter Summary

In this chapter, we revisited the established concept of block relative gain (BRG). The main contributions of this chapter include

(i) Extension of algebraic properties known for RGA to BRG.

(ii) Connection between the BRG and measures of block diagonal dominance, in particular Grosdidier's μ interaction measure [49] (see §4.4.3).

(iii) Correction and restatement of the common conjecture that a system is *weakly* interacting, if BRG is close to Identity (see §4.4.4).

Most of the results presented are based on steady state gain information only and are useful for controllability analysis and pairing selection. It is also shown that in some cases, steady state analysis may suggest inferior pairings. However, practical considerations justify its use, as in many cases, the only reliable information available at design stage is the steady state gain. Block decentralized controllers allow the designer to exploit a broader class of control structures that are not restricted to the two extremes of complete decentralization and complete centralization [83]. The pairing rules proposed in this paper will be helpful in bridging the gap between theory and practice of selection of block pairings.

4.A Scaled Gain Matrices for ALSTOM Gasifier System

The system is scaled such that $\|y_i\| \leq 1$ at all load conditions. The scaling matrix $\mathbf{X}$ is chosen such that $\mathbf{X}_{ii} = \max\{\|[\mathbf{G}_{100\%}(0)]_i\|_2, \|[\mathbf{G}_{50\%}(0)]_i\|_2, \|[\mathbf{G}_{0\%}(0)]_i\|_2\}$, where $[\mathbf{G}_{100\%}(0)]_i$ is the i^{th} row of the gain matrix at 100% load conditions. Then, $\mathbf{X} = \text{diag}(8.58 \times 10^5, 5.21 \times 10^4, 1.55 \times 10^4, 164.64)$ and the scaled gain matrices are obtained as $\mathbf{G}^s(0) = \mathbf{X}^{-1}\mathbf{G}(0)$.

$$\mathbf{G}^s_{100\%}(0) = \begin{bmatrix} 0.0385 & -0.0427 & 0.0444 & -0.0474 \\ -0.1115 & -0.0297 & 0.0770 & -0.0142 \\ 0.0327 & 0.8630 & 0.0477 & 0.5019 \\ 0.0088 & 0.1284 & -0.1101 & -0.2834 \end{bmatrix}$$

$$\mathbf{G}^s_{50\%}(0) = \begin{bmatrix} 0.0975 & -0.0381 & 0.0269 & -0.1130 \\ -0.2096 & -0.0500 & 0.1563 & -0.0211 \\ 0.0506 & 0.6923 & 0.0295 & 0.4200 \\ 0.0359 & 0.1804 & -0.1641 & -0.3967 \end{bmatrix}$$

$$\mathbf{G}^s_{0\%}(0) = \begin{bmatrix} 0.7938 & 0.1451 & -0.4361 & -0.3983 \\ -0.7641 & -0.1810 & 0.6161 & -0.0606 \\ 0.0958 & 0.3855 & -0.0301 & 0.2536 \\ 0.3119 & 0.3666 & -0.4841 & -0.7307 \end{bmatrix}$$

Chapter 5

Integrity of Systems under Decentralized Integral Control

A multivariate system has integrity if the block decentralized controller with integral action maintains closed loop stability in presence of possible controller failures. In this chapter, we show that the recently proposed necessary and sufficient conditions [52] for the system to possess integrity can be equivalently expressed in terms of well-known notions of block relative gain (BRG) [83] and Niederlinski index (NI) [87]. These results imply that the conditions based on BRG and NI, traditionally believed to be only necessary, are actually both necessary and sufficient. It is also shown that in general, establishing the existence of a fully decentralized controller with integral action such that the system has integrity is NP-hard.

5.1 Introduction

This chapter deals with reliable stabilization of stable linear systems using a decentralized controller with integral action in every channel. A system is said to possess integrity, if the closed loop stability is maintained with integral action in every output channel, when any combination of the individual controllers fails (see Definition 4.4). It is assumed that a controller that fails is immediately taken out of service, *i.e.* the corresponding entries in the block diagonal controller matrix are replaced by zero. Some researchers have considered the problem of checking whether the closed loop system is reliably stable for a given controller (see [13] for a review). The focus of this work is on deriving controller-independent conditions which can establish the existence or non-existence of a controller such that the system possess integrity.

With its practical implications, the problem of integrity against possible controller failures has been studied widely by researchers, particularly in the area of process control. For fully decentralized control, a well-known result that relates reliable stability with relative gain array (RGA) [17] is provided by Grosdidier *et. al.* [51]. It is shown that a system has integrity only if all the corresponding relative gains of the steady state gain matrix are positive. Similar to fully decentralized control, a system with specified block pairings has integrity only if the determinant of all the corresponding block relative gains (BRG) [83] of the steady state gain matrix are positive [50]. Grosdidier and Morari [49] generalized the concept of Niedrelinski index (NI) to block pairings to derive similar necessary conditions. Chiu and Arkun [30] have further suggested that the necessary conditions based on BRG and NI be evaluated for all principal block sub-matrices of the system. These necessary conditions based on BRG and NI are useful for eliminating alternatives for input-output pairings, as discussed in the previous chapter. It is not apparent whether the system with the pairings chosen based on these necessary conditions, will have integrity.

Recently, Gündes and Kabuli [52] presented necessary and sufficient conditions for assessing integrity of the system partitioned into 4 or less blocks. In this chapter, we show when the individual blocks are square, these conditions can be alternatively expressed in terms of BRG and NI. In general, these conditions do not guarantee that the block decentralized controller will have no unstable poles other than the origin, as is assumed in the derivation of necessary conditions based on NI and BRG. When the controllers are allowed to have any number of unstable poles, the alternative representation implies that the conditions based on BRG and NI, traditionally believed to be only necessary, are actually both necessary and sufficient. Since the expressions presented by Gündes and Kabuli [52] become increasingly complex with the number of blocks, an additional advantage of the alternative representation is that the extension to the general case, where the system is partitioned into any number of blocks, is relatively simple.

For fully decentralized control, we also show that the necessary and sufficient conditions due to Gündes and Kabuli [52] are satisfied iff a matrix, which depends on the system's steady state gain, is a $\mathcal{P}$-matrix. This observation suggests that establishing the existence of a fully decentralized controller with integral action such that the system has integrity is NP-hard unless P = NP [41].

5.2 Necessary and Sufficient Conditions

In this section, we present the necessary and sufficient conditions due to Gündes and Kabuli [52] such that the $\mathbf{G}(s)$ possess integrity. The discussion is limited to the case, where $\mathbf{G}(s)$ is partitioned into M non-overlapping square subsystems such that $\mathbf{G}_{ii}(0) \in \mathbb{R}^{m_i \times m_i}$; $i = 1, 2, \cdots, M$, $\sum_i m_i = n$. The block diagonal controller with integral action $\mathbf{K}(s)$ is expressed as $(1/s)\mathbf{C}(s)$, where $\mathbf{C}(s) = \operatorname{diag}(\mathbf{C}_{ii}(s))$ and $\mathbf{C}_{ii}(s)$ has same dimensions as $\mathbf{G}_{ii}(s)$ (see Figure 4.1). For notational convenience, $\mathbf{G}(0)$ is simply represented as $\mathbf{G}$.

To present the necessary and sufficient conditions for integrity of $\mathbf{G}(s)$, we need the following additional notation. For $j = 2, \cdots, M, i = 1, \cdots, j-1$, define

$$\mathbf{X}_{ij} = \mathbf{G}_{jj} - \mathbf{G}_{ji}\mathbf{G}_{ii}^{-1}\mathbf{G}_{ij} \tag{5.1}$$

When $M \geq 3$, for $k = 1, \cdots, M-2, \ell, m = k+1, \cdots, M, \ell \neq m$,

$$\mathbf{Y}_{\ell m}^{k} = \mathbf{G}_{\ell m} - \mathbf{G}_{\ell m}\mathbf{G}_{kk}^{-1}\mathbf{G}_{km} \tag{5.2}$$

and for $v = 3 \cdots M, q = 1, \cdots, v-2, r = q+1, \cdots, v-1$,

$$\mathbf{Z}_{rq}^{v} = \mathbf{X}_{qv} - \mathbf{Y}_{vr}^{q}\mathbf{X}_{qr}^{-1}\mathbf{Y}_{rv}^{q} \tag{5.3}$$

When $M = 4$, define

$$\mathbf{W} = \mathbf{Z}_{24}^{1} - (\mathbf{Y}_{43}^{1} - \mathbf{Y}_{42}^{1}\mathbf{X}_{12}^{-1}\mathbf{Y}_{23}^{1})(\mathbf{Z}_{23}^{1})^{-1}(\mathbf{Y}_{34}^{1} - \mathbf{Y}_{32}^{1}\mathbf{X}_{12}^{-1}\mathbf{Y}_{24}^{1}) \tag{5.4}$$

Theorem 5.1 Let $\mathbf{G}_{ii}$ be nonsingular for all $i = 1, \cdots, M$. There exists a block diagonal controller with integral action such that $\mathbf{G}(s)$ has integrity, if [52]

$$\det(\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1}) > 0 \tag{5.5}$$

for all $j = 2, \cdots, M, i = 1, \cdots, j-1$ and when $M \geq 3$

$$\det(\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{-1}) > 0 \tag{5.6}$$

for all $v = 3, \cdots, M, q = 1, \cdots, v-2, r = q+1, \cdots, v-1$ and when $M = 4$

$$\det(\mathbf{W}\mathbf{G}_{44}^{-1}) > 0 \tag{5.7}$$

Further, if any $M-1$ controllers are strictly proper, or when $\mathbf{G}_{ij}$ or $\mathbf{G}_{ji}, j = 2, \cdots, M, i = 1, \cdots, j-1$ are strictly proper or when any of these transfer matrices have real blocking zeros [109], (5.5)-(5.7) are also necessary.

The proof of Theorem 5.1 is quite long and requires elements from the coprime factorization theory [109]. As the proof provides no additional insight, the interested reader is referred to [52] for the proof of Theorem 5.1. Some remarks that are relevant to the the scope of this thesis are in order.

- The requirement that $\mathbf{G}_{ii}$ be nonsingular for all $i = 1, \cdots, M$ is necessary for existence of a controller with integral action such that the individual loops are stable. Consider that $\mathbf{G}_{ii}$ be singular for some $i = 1, \cdots, M$. Then, the loop transfer function $\mathbf{G}_{ii}\mathbf{K}_i = (1/s)\mathbf{G}_{ii}\mathbf{C}_i$ contains a hidden mode. Thus, the stabilization of the i^{th} loop is not possible and $\mathbf{G}(s)$ does not have integrity.

- Whereas the off-diagonal blocks of $\mathbf{G}(s)$ are not strictly proper or have real blocking zeros in general, the controllers can always be designed to be strictly proper. When all controllers are strictly proper, (5.5)-(5.7) are both necessary and sufficient for existence of a block diagonal controller with integral action such that $\mathbf{G}(s)$ has integrity. We recall that a similar assumption is made during the derivation the necessary conditions based on BRG and NI (see Theorems 4.1- 4.2).

- When the sufficient conditions (5.5)-(5.7) are satisfied, existence of a controller with integral action is guaranteed such the system has integrity. This controller, however, may have additional unstable poles other than at the origin of the complex plane. The existence of pure integral action controllers is guaranteed, when the more restrictive conditions: $\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1} \succ 0$, $\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{-1} \succ 0$ and $\mathbf{W}\mathbf{G}_{44}^{-1} \succ 0$, hold for all indices defined earlier.

- For fully decentralized control, $\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1} \succ 0$, $\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{-1} \succ 0$ and $\mathbf{W}\mathbf{G}_{44}^{-1} \succ 0$ is equivalent to (5.5)-(5.7). In this case, when (5.5)-(5.7) hold, existence of a pure integral action controller is guaranteed such that $\mathbf{G}(s)$ has integrity.

Gündes and Kabuli [52] have also presented a controller design method such that $\mathbf{G}(s)$ has integrity, when the sufficient conditions $\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1} \succ 0$, $\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{-1} \succ 0$ and $\mathbf{W}\mathbf{G}_{44}^{-1} \succ 0$ hold for all indices defined earlier. Generally, the positive-definiteness is defined only for symmetric matrices. By $\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1} \succ 0$, we imply that the symmetric part of $\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1}$, i.e. $\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1} + (\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1})^{*}$ is positive-definite.

5.3 Simplified Representation

In this section, we show that the conditions in Theorem 5.1 can be equivalently represented in terms of BRG and NI. For this purpose, we require evaluation of BRG and NI on the principal block sub-matrices of G. We define ψ as the ordered subset of first M integers consisting of at least 2 elements and Ψ as the ensemble of all such ψ. For example, when $M = 2$, $\Psi = \{(1,2)\}$ and $M = 3$, $\Psi = \{(1,2),(1,3),(2,3),(1,2,3)\}$.

Lemma 5.1 Let $\mathbf{G}_{ii}$ be nonsingular for all $i = 1, \cdots, M$. Then,

$$\begin{aligned}
\det(\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1}) &= \frac{\det(\mathbf{G}_{\{i,j\},\{i,j\}})}{\det(\mathbf{G}_{ii})\det(\mathbf{G}_{jj})} && (5.8)\\
\det(\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{-1}) \cdot \det(\mathbf{X}_{qr}\mathbf{G}_{rr}^{-1}) &= \frac{\det(\mathbf{G}_{\{q,r,v\},\{q,r,v\}})}{\det(\mathbf{G}_{qq})\det(\mathbf{G}_{rr})\det(\mathbf{G}_{vv})} && (5.9)\\
\det(\mathbf{W}\mathbf{G}_{44}^{-1}) \cdot \det(\mathbf{Z}_{23}^{1}\mathbf{G}_{33}^{-1}) \cdot \det(\mathbf{X}_{12}\mathbf{G}_{22}^{-1}) &= \frac{\det(\mathbf{G})}{\prod_{i=1}^{4}\det(\mathbf{G}_{ii})} && (5.10)
\end{aligned}$$

where $j = 2, \cdots, M, i = 1, \cdots, j-1$ and $v = 3, \cdots, M, q = 1, \cdots, v-2, r = q+1, \cdots, v-1$.

Proof: Since $\mathbf{G}_{ii}$ is nonsingular for all $i = 1, \cdots, M$, using (5.1),

$$\begin{aligned}
\det(\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1}) &= \det(\mathbf{I} - \mathbf{G}_{ji}\mathbf{G}_{ii}^{-1}\mathbf{G}_{ij}\mathbf{G}_{jj}^{-1})\\
&= \det\begin{pmatrix} \mathbf{I} & \mathbf{G}_{ji}\mathbf{G}_{ii}^{-1} \\ \mathbf{G}_{ij}\mathbf{G}_{jj}^{-1} & \mathbf{I} \end{pmatrix}\\
&= \det\begin{pmatrix} \mathbf{G}_{jj} & \mathbf{G}_{ji} \\ \mathbf{G}_{ij} & \mathbf{G}_{ii} \end{pmatrix} \det\begin{pmatrix} \mathbf{G}_{jj}^{-1} & 0 \\ 0 & \mathbf{G}_{ii}^{-1} \end{pmatrix}\\
&= \frac{\det(\mathbf{G}_{\{i,j\},\{i,j\}})}{\det(\mathbf{G}_{ii})\det(\mathbf{G}_{jj})}
\end{aligned}$$

where the second equality follows using Schur complement Lemma. The proofs of (5.9)-(5.10) require repeated use of Schur complement Lemma and are omitted for the sake of brevity. ■

Proposition 5.1 Let $\mathbf{G}_{ii}$ be nonsingular for all $i = 1, \cdots, M$. Then, the following are equivalent:

(1) $\det(\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1}) > 0 \quad \forall j = 2, \cdots, M, i = 1, \cdots, j-1$
$\det(\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{-1}) > 0 \quad \forall v = 3, \cdots, M, q = 1, \cdots, v-2, r = q+1, \cdots, v-1$
$\det(\mathbf{W}\mathbf{G}_{44}^{-1}) > 0$

(2) $\mathbf{NI}(\mathbf{G}_{\psi\psi}) > 0 \quad \forall \psi \in \Psi$ (5.11)

(3) $\det([\mathbf{\Lambda}_B(\mathbf{G}_{\psi\psi})]_{kk}) > 0 \quad \forall \psi \in \Psi, k = 1, \cdots, |\psi|$ (5.12)

where $|\cdot|$ denotes the cardinality of the set ψ.

Proof: We show that (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3), which implies (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

((1) $\Leftrightarrow$ (2)) Using (5.8), $\det(\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1}) > 0$ iff

$$\frac{\det(\mathbf{G}_{\{i,j\},\{i,j\}})}{(\det(\mathbf{G}_{ii})\det(\mathbf{G}_{jj}))} > 0$$

for all $j = 2, \cdots, M, i = 1, \cdots, j-1$. When $M \geq 3$, the ordered set $\{r, q\}$ is a subset of $\{i, j\}$. Then, $\det(\mathbf{X}_{qr}\mathbf{G}_{rr}^{-1}) > 0$ for all $v = 3, \cdots, M, q = 1, \cdots, v-2, r = q+1, \cdots, v-1$. Using (5.9), $\det(\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{-1}) > 0$, iff

$$\frac{\det(\mathbf{G}_{\{q,r,v\},\{q,r,v\}})}{(\det(\mathbf{G}_{qq})\det(\mathbf{G}_{rr})\det(\mathbf{G}_{vv}))} > 0$$

for all $v = 3, \cdots, M, q = 1, \cdots, v-2, r = q+1, \cdots, v-1$. Similarly, when $M = 4$, $\det(\mathbf{W}\mathbf{G}_{44}^{-1}) > 0$, iff

$$\frac{\det(\mathbf{G})}{(\prod_{i=1}^{4}\det(\mathbf{G}_{ii}))}$$

The necessity and sufficiency of (5.11) follows by combining all these arguments and noting that $\Psi = \{i, j\} \bigcup \{q, r, v\}$.

((2) $\Leftrightarrow$ (3)), Using (5.8),

$$\mathrm{NI}(\mathbf{G}_{\{i,j\},\{i,j\}}) = \frac{\det(\mathbf{G}_{\{i,j\},\{i,j\}})}{\det(\mathbf{G}_{ii})\det(\mathbf{G}_{jj})} = \det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j\},\{i,j\}})]_{ii}) \quad \forall i, j \leq M, i \neq j$$

Then, $\mathrm{NI}(\mathbf{G}_{\{i,j\},\{i,j\}}) > 0$, iff $\det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j\},\{i,j\}})]_{ii}) > 0$ for all $i, j \leq M, i \neq j$. When, $M \geq 3$, using (5.9),

$$\begin{aligned}
\mathrm{NI}(\mathbf{G}_{\{i,j,k\},\{i,j,k\}}) &= \frac{\det(\mathbf{G}_{\{i,j,k\},\{i,j,k\}})}{\det(\mathbf{G}_{ii})\det(\mathbf{G}_{jj})\det(\mathbf{G}_{kk})} \\
&= \frac{\det(\mathbf{G}_{\{i,j,k\},\{i,j,k\}})}{\det(\mathbf{G}_{kk})\det(\mathbf{G}_{\{i,j\},\{i,j\}})} \frac{\det(\mathbf{G}_{\{i,j\},\{i,j\}})}{\det(\mathbf{G}_{ii})\det(\mathbf{G}_{jj})} \\
&= \frac{\mathrm{NI}(\mathbf{G}_{\{i,j\},\{i,j\}})}{\det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j,k\},\{i,j,k\}})]_{kk})} \quad \forall i, j, k \leq M, i \neq j \neq k
\end{aligned}$$

Since $\mathrm{NI}(\mathbf{G}_{\{i,j\},\{i,j\}}) > 0$ for all $i, j \leq M, i \neq j$, $\mathrm{NI}(\mathbf{G}_{\{i,j,k\},\{i,j,k\}}) > 0$, iff $\det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j,k\},\{i,j,k\}})]_{ii}) > 0$ for all $i, j, k \leq M, i \neq j \neq k$. When, $M = 4$, using (5.10) and similar arguments as above,

$$\mathrm{NI}(\mathbf{G}) = \frac{\mathrm{NI}(\mathbf{G}_{\{i,j,k\},\{i,j,k\}})}{\det([\mathbf{\Lambda}_B(\mathbf{G})]_{\ell\ell})} \quad \forall i, j, k \leq M, i \neq j \neq k, \ell = \{1, \cdots, M\}/\{i, j, k\}$$

Since $\text{NI}(\mathbf{G}_{\{i,j,k\},\{i,j,k\}}) > 0$, $\text{NI}(\mathbf{G}) > 0$ iff $\det([\mathbf{\Lambda}_B(\mathbf{G})]_{\ell\ell}) > 0$ for all $i, j, k \leq M, i \neq j \neq k, \ell = \{1, \cdots, M\}/\{i, j, k\}$. Now, the necessity and sufficiency of (5.12) follows by combining all these arguments. ■

To check whether (5.11) or (5.12) hold, one needs to calculate NI or BRG for all principal sub-matrices of $\mathbf{G}$ that can be formed by combining elements of the diagonal blocks and the corresponding off-diagonal blocks. A similar method was earlier considered by Chiu and Arkun [30], where (5.11) and (5.12) were shown to be necessary under the assumptions that $\mathbf{G}(s)\mathbf{C}(s)$ is strictly proper and $\mathbf{C}(s)$ is stable.

Proposition 5.1 implies that (5.5)-(5.7) are satisfied iff (5.11) or (5.12) hold. Then, similar to Theorem 5.1, (5.11) and (5.12) are both necessary and sufficient, when $\mathbf{C}(s)$ is restricted to be strictly proper. As pointed out earlier, satisfying (5.5)-(5.7) is equivalent to satisfying $\mathbf{X}_{ij}\mathbf{G}_{jj}^{-1} \succ 0$, $\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{-1} \succ 0$ and $\mathbf{W}\mathbf{G}_{44}^{-1} \succ 0$ for fully decentralized control. Thus, the existence of a stable $\mathbf{C}(s)$ is guaranteed such that $\mathbf{G}(s)$ has integrity for fully decentralized control, but in general, there may not exist a stable $\mathbf{C}(s)$ such that $\mathbf{G}(s)$ has integrity, even when (5.11) or (5.12) hold. It is worth pointing out the requirement that $\mathbf{C}(s)$ be stable is restrictive, as noted by Campo and Morari [18], but is practically relevant. Derivation of necessary and sufficient conditions for $\mathbf{G}(s)$ to possess integrity such that that $\mathbf{C}(s)$ is stable remains an issue for future work.

As M increases, the expressions presented by Gündes and Kabuli [52] become increasingly complex (*cf.* (5.5)-(5.7)). On the other hand, the extension to the general case is simple (by induction), when the conditions are expressed in terms of BRG or NI. In this chapter, we have only dealt with the case, where $\mathbf{G}_{ii}$ are square. The results of Gündes and Kabuli [52] also hold when the individual blocks are possibly non-square with every loop having more inputs than outputs for integral action. In this case, the conditions remain the same, except (5.1)-(5.4) need to be modified to accommodate the right inverses of different non-square sub-matrices of $\mathbf{G}$. Similar to the proofs of Lemma 5.1 and Proposition 5.1, it can be shown that (5.5)-(5.7) holds for non-square blocks, iff

$$\det([\mathbf{G}\mathbf{G}_{bd}^{\dagger}]_{\psi\psi}) > 0 \quad \forall \psi \in \Psi \tag{5.13}$$

where $\mathbf{G}_{bd} = \text{diag}(\mathbf{G}_{ii})$ and † denotes some right inverse. Note that $\mathbf{G}\mathbf{G}_{bd}^{\dagger}$ can be treated as the generalized Neidrilinski index, where the individual blocks are non-square [49].

To verify whether (5.11) holds, NI needs to be evaluated exactly $2^M - (M+1)$ times, whereas verification of (5.12) requires that BRG be evaluated many more times. This ambiguity is explained by noting that evaluation of BRG for all principal block sub-matrices of $\mathbf{G}$ is not necessary. For example, when $M = 3$,

$$\frac{\det([\mathbf{\Lambda}_B(\mathbf{G})]_{jj})\det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,k\},\{i,k\}})]_{ii})}{\det([\mathbf{\Lambda}_B(\mathbf{G})]_{kk})} = \det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j\},\{i,j\}})]_{ii})$$

Thus, if all the terms on the LHS of the above expression are positive, $\det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j\},\{i,j\}})]_{ii})$ is always positive. The task of finding the set of $2^M - (M+1)$ non-redundant BRGs requires some book-keeping. In this sense, the use of (5.11) is advantageous over the use of (5.12).

5.4 Computational Complexity

In this section, we present some results on computational complexity for establishing the existence of a block diagonal controller such that $\mathbf{G}(s)$ has integrity. It is shown that this problem is NP-hard, unless P = NP [41]. We introduce the useful notion of $\mathcal{P}$-matrices, which form the basis of the proof for NP-hardness.

Definition 5.1 A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called a $\mathcal{P}$-matrix, if all the principal minors of $\mathbf{A}$ are positive [63].

Lemma 5.2 Let $\mathbf{G}_{bd}$ be a non-singular matrix consisting of the diagonal elements of $\mathbf{G}$. Then, (5.5)-(5.7) are satisfied for all the indices defined in Theorem 5.1, iff $\mathbf{G}\mathbf{G}_{bd}^{-1}$ is a $\mathcal{P}$-matrix.

Proof: It follows from Proposition 5.1 that (5.5)-(5.7) are satisfied for all the indices defined in Theorem 5.1 iff (5.11) holds. Note that $\text{NI}(\mathbf{G}_{\psi\psi}) = \det([\mathbf{G}\mathbf{G}_{bd}^{-1}]_{\psi\psi})$ for all $\psi \in \Psi$ and $[\mathbf{G}\mathbf{G}_{bd}^{-1}]_{ii} = 1$ for all $i = 1, \cdots, M$. Then, $\text{NI}(\mathbf{G}_{\psi\psi}) > 0$ for all $\psi \in \Psi$, iff $\mathbf{G}\mathbf{G}_{bd}^{-1}$ is $\mathcal{P}$-matrix. ■

Proposition 5.2 Let $\mathbf{G}_{bd}$ be a non-singular matrix consisting of the diagonal elements of $\mathbf{G}$. If the controller $\mathbf{K}(s)$ is restricted to be strictly proper, the problem of establishing the existence of a diagonal controller such that $\mathbf{G}(s)$ has integrity is NP-hard, unless P = NP.

Proof: When the controller $\mathbf{K}(s)$ is restricted to be strictly proper and $M \leq 4$, satisfying (5.5)-(5.7) for all the indices defined in Theorem 5.1 is necessary and sufficient for the problem of establishing the existence of a diagonal controller such that $\mathbf{G}(s)$ has integrity. Similar conditions can be derived using Proposition 5.1 and induction, when M is arbitrary. Lemma 5.2 shows that these conditions hold iff $\mathbf{G}\mathbf{G}_{bd}^{-1}$ is $\mathcal{P}$-matrix or $\det([\mathbf{G}\mathbf{G}_{bd}^{-1}]_{\psi\psi})$

for all $\psi \in \Psi$. Note that the transformation of (5.5)-(5.7) to $\det([\mathbf{G}\mathbf{G}_{bd}^{-1}]_{\psi\psi})$ for all $\psi \in \Psi$ requires only elementary operations and can be completed in polynomial time. The result follows by noting that verifying whether a given matrix is $\mathcal{P}$-matrix is co-NP-complete [31]. ■

As pointed out earlier, for fully decentralized control, satisfying (5.5)-(5.7) guarantees the existence of a pure integral action controller such that $\mathbf{G}(s)$ has integrity. In this case, the problem of establishing the existence of a diagonal controller such that $\mathbf{G}(s)$ has integrity remains NP-hard, when the controllers are further restricted to have poles at origin only. Similar conclusions can also be drawn using (5.13) for the case, when the individual blocks of $\mathbf{G}$ are non-square, but have a single output only.

The NP-hardness of the integrity problem suggests that as M increases, there exists systems, whose integrity cannot be verified in polynomial time. For particular instances of the problem, it may still be possible to establish the existence of the diagonal controller such that $\mathbf{G}(s)$ has integrity in polynomial time. The time complexity of an algorithm evaluating all the principal minors of the given real matrix is approximately $\mathcal{O}(n^3 2^n)$. Tsatsomeros and Li [105] have presented a recursive algorithm that reduces the time complexity to $\mathcal{O}(2^n)$. Recently, Rump [95] has proposed an algorithm, whose time complexity is not necessarily exponential, but can be exponential in the worst case. Rump [95] has applied this algorithm to a test set of parameterized matrices, whose membership in the class of $\mathcal{P}$-matrices is known beforehand for the given value of the parameter. It is shown that the algorithm can successfully verify whether these matrices having dimensions up to 100×100 are $\mathcal{P}$-matrices in polynomial time.

When the controller is block decentralized, one only needs to check the positiveness of minors of the sub-matrices of $\mathbf{G}\mathbf{G}_{bd}^{-1}$ that can be formed by combining elements of different blocks and the corresponding off-block diagonal elements. In this case, if $\det([\mathbf{G}\mathbf{G}_{bd}^{-1}]_{\psi\psi}) > 0$ for all $\psi \in \Psi$, we call $\mathbf{G}\mathbf{G}_{bd}^{-1}$ a block $\mathcal{P}$-matrix in the spirit of $\mathcal{P}$-matrices. It is conjectured that under the same conditions as Proposition 5.2, establishing the existence of the block diagonal controller such that $\mathbf{G}(s)$ has integrity is also NP-hard. The algorithm of Tsatsomeros and Li [105] is based on Schur complement lemma and is easily extended for verifying block $\mathcal{P}$-matrices. It is not clear at present, if it is possible to use the algorithm of Rump [95] for block matrices. We next present a sufficient condition for verifying whether $\mathbf{G}\mathbf{G}_{bd}^{-1}$ is a $\mathcal{P}$- or block $\mathcal{P}$-matrix.

Proposition 5.3 Let $\mathbf{G}_{bd} = \mathrm{diag}(\mathbf{G}_{ii})$, where $\mathbf{G}_{ii} \in \mathbb{R}^{m_i \times m_i}$, $i = 1, \cdots, M$ and $\mathbf{G}_{bd}$ is non-singular. Define $\mathbf{E} = (\mathbf{G} - \mathbf{G}_{bd})\mathbf{G}_{bd}^{-1}$. Then, $\mathbf{G}\mathbf{G}_{bd}^{-1}$ is block $\mathcal{P}$-matrix wrt the structure of $\mathbf{G}_{bd}$, if $\det(\mathbf{I}+0.5\mathbf{E}) \neq 0$ and

$$\mu_{\boldsymbol{\Delta}}((\mathbf{I}+0.5\mathbf{E})^{-1}\mathbf{E}) < 2 \tag{5.14}$$

where $\boldsymbol{\Delta} = \{\mathrm{diag}(\delta_i \cdot \mathbf{I}_{m_i}), \delta_i \in \mathbb{C}, |\delta_i| \leq 1, i = 1, \cdots, M\}$.

Proof: Note that $\mathbf{G}\mathbf{G}_{bd}^{-1} = \mathbf{I}+\mathbf{E}$. Define, $\boldsymbol{\Delta}_1 = \{\mathrm{diag}(\epsilon_i \cdot \mathbf{I}_{m_i}), \epsilon_i = \{0,1\}, i = 1, \cdots, M\}$. Then, $\mathbf{G}\mathbf{G}_{bd}^{-1}$ is a block $\mathcal{P}$-matrix iff,

$$\det(\mathbf{I}+\mathbf{E}\tilde{\boldsymbol{\Delta}}_1) > 0 \quad \forall \tilde{\boldsymbol{\Delta}}_1 \in \boldsymbol{\Delta}_1 \tag{5.15}$$

Further, defining $\boldsymbol{\Delta}_2 = \{\mathrm{diag}(\epsilon_i \cdot \mathbf{I}_{m_i}), \epsilon_i \in \mathbb{C}, |\epsilon_i| \leq 1, i = 1, \cdots, M\}$ and noting that $\boldsymbol{\Delta}_1 \subset \boldsymbol{\Delta}_2$, (5.15) holds if,

$$\det(\mathbf{I}+\mathbf{E}\tilde{\boldsymbol{\Delta}}_2) > 0 \quad \forall \tilde{\boldsymbol{\Delta}}_2 \in \boldsymbol{\Delta}_2$$

The determinant is a continuous function over convex sets. Thus, if $\det(\mathbf{I}+\mathbf{E}\tilde{\boldsymbol{\Delta}}_2)$ changes sign over the set $\boldsymbol{\Delta}_2$, there exists some $\tilde{\boldsymbol{\Delta}}_2 \in \boldsymbol{\Delta}_2$ such that $\det(\mathbf{I}+\mathbf{E}\tilde{\boldsymbol{\Delta}}_2) = 0$. Since, $\boldsymbol{\Delta}_1 \subset \boldsymbol{\Delta}_2$, (5.15) holds if,

$$\begin{aligned} \det(\mathbf{I}+\mathbf{E}\tilde{\boldsymbol{\Delta}}_2) &\neq 0 \quad \forall \tilde{\boldsymbol{\Delta}}_2 \in \boldsymbol{\Delta}_2 \\ \Leftrightarrow \quad \mu_{\boldsymbol{\Delta}_2}(\mathbf{E}) &< 1 \end{aligned} \tag{5.16}$$

The inequality (5.16) is conservative as $\mathbf{I}, -\mathbf{I} \in \boldsymbol{\Delta}_2$. To reduce conservativeness [11, 13], for every $\tilde{\boldsymbol{\Delta}} \in \boldsymbol{\Delta}$, $\tilde{\boldsymbol{\Delta}}_2 \in \boldsymbol{\Delta}_2$, define $\tilde{\boldsymbol{\Delta}}_2 = 0.5(\mathbf{I}+\tilde{\boldsymbol{\Delta}})$. Then,

$$\begin{aligned} \det(\mathbf{I}+\mathbf{E}\tilde{\boldsymbol{\Delta}}_2) &= \det(\mathbf{I}+0.5\mathbf{E}+0.5\mathbf{E}\tilde{\boldsymbol{\Delta}}) \\ &= \det(\mathbf{I}+0.5\mathbf{E})\det(\mathbf{I}+0.5(\mathbf{I}+0.5\mathbf{E})^{-1}\mathbf{E}\tilde{\boldsymbol{\Delta}}) \end{aligned}$$

When (5.14) holds, $\det(\mathbf{I}+0.5(\mathbf{I}+0.5\mathbf{E})^{-1}\mathbf{E}\tilde{\boldsymbol{\Delta}})$ does not change sign over the set $\boldsymbol{\Delta}$ and $\mathbf{G}\mathbf{G}_{bd}^{-1}$ is block $\mathcal{P}$-matrix wrt the structure of $\mathbf{G}_{bd}$. ■

The sub-matrices of positive-definite are also positive-definite [62]. Thus, when $\mathbf{G}\mathbf{G}_{bd}^{-1} \succ 0$, $\mathbf{G}\mathbf{G}_{bd}^{-1}$ is $\mathcal{P}$ and thus block $\mathcal{P}$-matrix. Proposition 5.3 is less conservative than this sufficient condition, as the controller structure is taken into account. Proposition 5.3 is still conservative, as $\tilde{\boldsymbol{\Delta}}_1$ is a strict subset of $\tilde{\boldsymbol{\Delta}}$. A practical approach is to check if (5.14) holds and if not, use the algorithm of Tsatsomeros and Li [105] for block decentralized control or Rump [95] for fully decentralized control.

5.5 Chapter Summary

In this chapter, we presented the necessary and sufficient conditions due to Gündes and Kabuli [52] for establishing the existence of a decentralized controller such that the system partitioned into 4 or less blocks has integrity. It is shown that these conditions can be alternately represented in terms of block relative gain (BRG) and Niedrilinski index (NI). The following results are shown using the alternate representation:

- The conditions due to Gündes and Kabuli [52] can be easily generalized to the case, when the system is partitioned into arbitrary number of blocks.
- When the controller is allowed to have unstable poles other than at the origin, the conditions based on BRG and NI, traditionally believed to be only necessary, are in fact both necessary and sufficient. For fully decentralized control, the additional assumption of the controller having unstable poles other than at origin is not required.
- The problem of establishing the existence of the diagonal controller such that the system has integrity is equivalent to verifying whether a given real matrix is a $\mathcal{P}$-matrix, which is co-NP-complete.

Though the integrity problem for fully decentralized control is shown to be NP-hard, it may be possible to solve particular instances of this problem using the algorithm of Rump [95]. It is conjectured that the integrity problem for block decentralized control is also NP-hard. A (conservative) sufficient condition is proposed for establishing the existence of the block diagonal controller such that the system has integrity. Future work will focus on extending the algorithm of Rump [95] to the block $\mathcal{P}$-matrix case and determination of necessary and sufficient conditions for integrity, when the controller is restricted to have poles only at the origin.

Chapter 6

Decentralized Minimum Variance Benchmark

This chapter deals with performance assessment of decentralized controllers using the minimum variance (MV) benchmark. The available MV benchmarks do not take the structure of the controller into account and can give overly optimistic estimates of achievable performance, when applied to systems under decentralized control. We propose an approximate solution to this problem obtained by explicitly solving simple linear matrix equations. As a special case of this general result, we also present an upper bound on the achievable performance for systems under multi-loop PID control. These results are useful for assessing the feasibility of significant performance improvement by re-tuning of the decentralized controller and input-output pairing selection [1].

6.1 Introduction

In the control literature, it is common to represent a non-linear, time-varying process by a LTI model and design a controller based on this. In the presence of changing operating conditions and disturbance dynamics, the closed loop performance of the controller designed based on this approximation may deteriorate over time. Sustained benefits can be reaped by monitoring the performance and taking appropriate corrective actions, in the case of large deviations from the designed performance.

Poor controller tuning is one of the primary reasons for performance deterioration of industrial controllers. It is important to assess the feasibility of significant performance improvement, before the task of controller tuning is undertaken. This purpose is well served by the minimum variance (MV) benchmark, where the controller objective is defined in terms of output variance. The MV benchmark represents the theoretical lower bound on the achievable output variance. The output variance can be reduced by controller tuning, when the actual variance differs significantly from the MV benchmark; otherwise, different approaches should be considered *e.g.* the use of feedforward controller or additional manipulated variables.

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[1] A preliminary version of this chapter was presented at 53rd conference of Canadian Society of Chemical Engineers, Hamilton, ON, 2003

The idea of MV control was introduced by Åström [5]. It was shown that the time series representation of the closed loop expression from the disturbances to the outputs can be partitioned into controller invariant and controller dependent parts. The MV control law is found by setting the controller dependent part to zero and the variance contribution of the controller invariant part represents the lower bound on the achievable performance (defined in terms of variability of outputs).

Harris [53] showed that with *a priori* knowledge of time delay, MV benchmark can be estimated using routine closed loop operating data and established it as a tool for performance monitoring of SISO systems. This approach is further extended to MIMO systems by Harris *et al.* [54] and Huang *et al.* [70]. Qin [91] and Harris *et al.* [55] provide comprehensive reviews of MV based and other performance assessment tools.

Though useful, the available MV benchmark shows limitations, when applied to systems using (block) decentralized or multi-loop control. The conventional approaches towards performance assessment of such controllers include:

- Loop by loop analysis
- Use of the MV benchmark for full multivariate controllers

The MV benchmark fails to take the process interactions into account, when applied in a loop-wise fashion; whereas, the full multivariable benchmark assumes more degrees of freedom for performance improvement than are available in the actual controller. In either case, the bound on the achievable output variance is loose and can be overly optimistic. In many cases, it may lead the practicing engineer to search for the non-existent decentralized controller to match the performance of the MV benchmark. The gap between the benchmark and achievable performance further increases when the decentralized controller is restricted to be of reduced complexity, *e.g.* proportional integral derivative (PID) controller [75]. Thus, a decentralized MV benchmark is required, which takes the controller structure into account. These arguments are further illustrated using the following example adapted from Huang and Shah [69]:

Example 6.1 Consider $\mathbf{y}(t) = \mathbf{G}(q^{-1})\mathbf{u}(t) + \mathbf{G}_w(q^{-1})\mathbf{a}(t)$, where q^{-1} is the backshift operator, $\mathbf{a}(t)$ is Gaussian noise with unit variance and

$$\mathbf{G} = \begin{bmatrix} \frac{q^{-2}}{1-0.4q^{-1}} & \frac{2q^{-2}}{1-0.5q^{-1}} \\ \frac{q^{-2}}{1-0.1q^{-1}} & \frac{q^{-2}}{1-0.2q^{-1}} \end{bmatrix} \quad \mathbf{G}_w = \begin{bmatrix} \frac{2}{1-0.9q^{-1}} & \frac{1}{1-0.3q^{-1}} \\ \frac{1}{1-0.4q^{-1}} & \frac{2}{1-0.5q^{-1}} \end{bmatrix}$$

The objective is to assess the performance of a multi-loop controller of the form $k\mathbf{I}$, $k = 0.17$. Under closed loop control, $E[\mathrm{tr}(\mathbf{y}(t)\mathbf{y}(t)^T] = 23.65$, where $E[.]$ is the expectation operator. The MV benchmark for full multivariate controller is 14.5, but no k or a dynamic compensator could be found that matches this benchmark closely. As shown later, the given controller structure inherently limits the achievable performance and the controller $0.17\mathbf{I}$ is nearly optimal for the given controller structure.

An explicit solution to the decentralized MV control problem has great theoretical and practical value, but is equally difficult to realize. The primary difficulty lies in enforcing the decentralized structure on the controller, as this yields a non-convex optimization problem [103]. Yuz and Goodwin [114] have suggested a two-step approach for determining an upper bound on the achievable output variance using a decentralized controller:

- A decentralized controller is designed based on only the diagonal elements of the system.
- The controller is redesigned to compensate for the ignored off-diagonal elements using an approximation of the sensitivity function.

Though the initial design based on the diagonal elements accommodates the controller structure, the controller redesign step requires some care and numerical search. Further, the utility of the method in its present form is limited to step disturbances only.

In this paper, we take a fundamentally different approach to derive an approximate solution for the decentralized MV control problem. The controller structure is posed as a constraint on the optimization problem and a suboptimal solution is obtained by explicitly solving the linear matrix equations defining the stationary point. As a special case, we present an upper bound on the achievable output variance for systems under multi-loop PID control. The results presented here do not require controller redesign [114] or numerical search [75]; however the simplicity of the result comes at the cost of sub-optimality. These results are useful for various purposes:

1. Performance assessment of existing decentralized or multi-loop controllers.
2. Selection of input-output pairings based on achievable decentralized performance.
3. Providing a good initial guess for non-convex parameter search methods.

6.2 Interactor Matrices

Before proceeding with the main development, we present the useful concept of interactor matrices introduced in [69].

Definition 6.1 For every $n_1 \times n_2$ proper, rational polynomial transfer matrix $\mathbf{G}(q^{-1})$, there is a unique, non-singular, $n_1 \times n_1$ lower triangular polynomial matrix $\mathbf{D}(q)$, such that $|\mathbf{D}(q)| = q^r$ and [45]

$$\lim_{q^{-1}\to 0} \mathbf{D}(q)\mathbf{G}(q^{-1}) = \lim_{q^{-1}\to 0} \tilde{\mathbf{G}}(q^{-1}) = \tilde{\mathbf{G}}(0) \tag{6.1}$$

where $\tilde{\mathbf{G}}(0)$ is a full rank constant matrix [69]. The matrix $\mathbf{D}(q)$ is called the *interactor matrix.*

For univariate systems, the MV benchmark primarily depends on the time delay associated with $\mathbf{G}(q^{-1})$ [5]. This time delay can also be interpreted as the non-invertible part of the transfer matrix, as its inverse is non-causal. Similarly, the multivariate system $\mathbf{G}(q^{-1})$ can be factored as $\mathbf{G}(q^{-1}) = \mathbf{D}^{-1}(q^{-1})\tilde{\mathbf{G}}(q^{-1})$ such that $\tilde{\mathbf{G}}(q^{-1})$ and $\mathbf{D}^{-1}(q^{-1})$ contain the invertible and non-invertible parts of $\mathbf{G}(q^{-1})$ respectively. The interactor matrix generalizes the time delay for univariate systems to the multivariate case [69] and can be written as,

$$\mathbf{D}(q) = \mathbf{D}_0(q)q^d + \mathbf{D}_1(q)q^{d-1} + \cdots \mathbf{D}_{d-1}(q)q$$

where d denotes the order of the interactor matrix.

When $\mathbf{D}(q)$ assumes the form $\mathbf{D}(q) = q^d\mathbf{I}$, $\mathbf{D}(q)$ is called a simple interactor matrix. Similarly, an interactor matrix with the form $\mathbf{D}(q) = \text{diag}(q^{d_1}, \cdots, q^{d_n})$ is called a diagonal interactor matrix. $\mathbf{D}(q)$ with no special structure is called a general interactor matrix.

The lower triangular form is only one of the possible realizations of the interactor matrices. In general, the interactor matrix can also be upper triangular or a full matrix. One realization of the interactor matrix that is of immediate interest to us, is when $\mathbf{D}(q)$ is unitary.

Definition 6.2 For a rational proper, transfer matrix $\mathbf{G}(q^{-1})$ having full rank, let the $\mathbf{D}(q)$ satisfying (6.1) also satisfies $\mathbf{D}^T(q^{-1})\mathbf{D}(q) = \mathbf{I}$. Then, $\mathbf{D}(q)$ is called a *unitary interactor matrix* [89].

The unitary interactor matrix is non-unique, but two unitary interactor matrices are related by transformation through a unitary matrix [69]. The unitary interactor matrix is useful for deriving the MV control law, when every output are given equal importance. Huang and Shah [68] have introduced the concept of weighted unitary matrices to handle the cases, where individual outputs have different importance in the control objective.

6.3 Problem Formulation

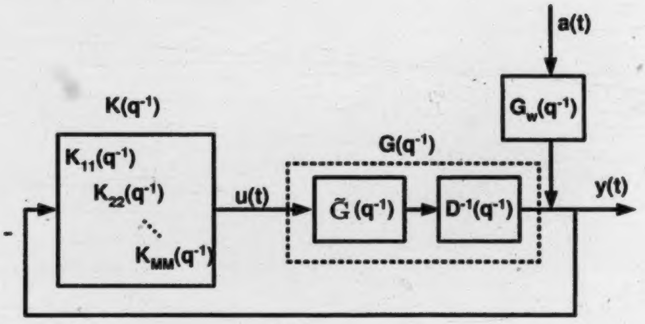

Figure 6.1: Separation of interactor matrix

Consider the system shown in Figure 6.1, where $\mathbf{K}(q^{-1}) = \text{diag}(\mathbf{K}_{ii}(q^{-1}))$, $i = 1, \cdots, M$. The objective is to find a controller such that the variance of $\mathbf{y}(t)$ or $E[\text{tr}(\mathbf{y}(t)\mathbf{y}(t)^T]$ is minimized. We make the following simplifying assumptions:

1. $\mathbf{G}(q^{-1})$ and $\mathbf{G}_w(q^{-1})$ are stable, causal transfer matrices, contain no zeros outside the unit circle and are square having dimensions $n \times n$.
2. $\mathbf{a}(t)$ is a random noise sequence with unit variance and $\mathbf{y}(t)$ is stationary up to its second moment.

The assumption that $\mathbf{G}(q^{-1})$ and $\mathbf{G}_w(q^{-1})$ are square is made for notational simplicity and can easily be relaxed for generalization purposes. When $\mathbf{G}_w$ contains zeros outside the unit circle, these zeros can be factored through an all pass factor factorization without affecting the noise spectrum [69]. Further, there is no loss of generality in assuming that the system is affected by noise having unit variance. When $E[\mathbf{a}(t)\mathbf{a}^T(t)] \neq \mathbf{I}$, the noise model can always be scaled to satisfy this assumption.

Next, we formulate the optimization problem that can be solved to obtain the solution to the decentralized MV control problem. In the remaining discussion, the arguments q^{-1} and t are dropped for ease of representation. Let the system shown in Figure 6.1 be expressed as

$$\begin{aligned} \mathbf{y} &= \mathbf{D}^{-1}\tilde{\mathbf{G}}\mathbf{u} + \mathbf{G}_w\mathbf{a} \\ \text{or } \mathbf{D}_1\mathbf{y} &= q^{-d}\tilde{\mathbf{G}}\mathbf{u} + \bar{\mathbf{G}}_d\mathbf{a} \end{aligned} \tag{6.2}$$

where $\mathbf{D}_1 = q^{-d}\mathbf{D}$, $\bar{\mathbf{G}}_d = \mathbf{D}_1\mathbf{G}_w$ and d is the order or number of non-zero impulse response matrices of $\mathbf{D}$. Using Diophantine's identity, $\bar{\mathbf{G}}_d = \bar{\mathbf{F}} + q^{-d}\bar{\mathbf{R}}$ and $\mathbf{u} = -\mathbf{K}\mathbf{y}$ for regulatory control,

$$\mathbf{D}_1\mathbf{y} = -q^{-d}\tilde{\mathbf{G}}\mathbf{K}\mathbf{y} + (\bar{\mathbf{F}} + q^{-d}\bar{\mathbf{R}})\mathbf{a} \tag{6.3}$$

Using (6.2), $\mathbf{a} = \bar{\mathbf{G}}_d^{-1}(\mathbf{D}_1\mathbf{y} - q^{-d}\tilde{\mathbf{G}}\mathbf{u})$. With simple algebraic manipulations, (6.3) can be simplified as,

$$\mathbf{D}_1\mathbf{y} = \bar{\mathbf{F}}\mathbf{a} + q^{-d}(\bar{\mathbf{R}}\mathbf{G}_w^{-1} - \bar{\mathbf{F}}\bar{\mathbf{G}}_w^{-1}\tilde{\mathbf{G}}\mathbf{K})\mathbf{y} \tag{6.4}$$

Since $E[\text{tr}(\mathbf{y}(t)\mathbf{y}(t)^T)] = E[\text{tr}(\mathbf{D}_1\mathbf{y}(t)\mathbf{y}(t)^T\mathbf{D}_1^T)]$ [69, Lemma 4.3.1] and $\bar{\mathbf{F}}$ is controller invariant, the second term in (6.4) can be set to zero to obtain the full multivariable MV control law. When the controller has structural constraints, this may not be possible since $\mathbf{K}$ has fewer degrees of freedom than the full multivariable controller.

Let $\mathbf{A} = \bar{\mathbf{R}}\mathbf{G}_w^{-1}$, $\mathbf{B} = \bar{\mathbf{F}}\bar{\mathbf{G}}_d^{-1}\tilde{\mathbf{G}}$ and $\mathbf{L} = \mathbf{A} - \mathbf{B}\mathbf{K}$. Then using (6.4),

$$\begin{aligned} \mathbf{y} &= (\mathbf{D}_1 - q^{-d}\mathbf{L})^{-1}\bar{\mathbf{F}}\mathbf{a} \\ &= (\mathbf{I} - q^{-d}\mathbf{D}_1^T\mathbf{L})^{-1}\mathbf{D}_1^T\bar{\mathbf{F}}\mathbf{a} \end{aligned}$$

When the spectral radius of $\mathbf{D}_1^T\mathbf{L}(e^{j\omega})$ is less than 1 for all $\omega = [0\ ,\ 2\pi]$ or the closed loop system is stable, the series expansion of $(\mathbf{I} - q^{-d}\mathbf{D}_1^T\mathbf{L})^{-1}$ is convergent. Thus,

$$\mathbf{y} = \left(\sum_{i=0}^{\infty}(q^{-d}\mathbf{D}_1^T\mathbf{L})^i\right)\mathbf{D}_1^T\bar{\mathbf{F}}\mathbf{a} \tag{6.5}$$

Since $E[\mathbf{a}(t)\mathbf{a}^T(t+\tau)] = 0$ for all $\tau \neq 0$ and $\mathbf{D}_1$ is a unitary transfer matrix,

$$\begin{aligned} E[\text{tr}(\mathbf{y}\mathbf{y}^T)] &= \|\mathbf{D}_1^T\bar{\mathbf{F}}\|_2^2 + \|\mathbf{D}_1^T\mathbf{L}\mathbf{D}_1^T\bar{\mathbf{F}}\|_2^2 + \cdots \\ &= \|\bar{\mathbf{F}}\|_2^2 + \|\mathbf{L}\mathbf{D}_1^T\bar{\mathbf{F}}\|_2^2 + \cdots \end{aligned} \tag{6.6}$$

The higher order terms on the RHS of (6.6) are non-linear in $\mathbf{K}$. An approximate solution to the decentralized MV control problem is obtained by ignoring these terms and finding the stationary point of $\|\mathbf{L}\mathbf{D}_1^T\bar{\mathbf{F}}\|_2^2$ wrt block diagonal $\mathbf{K}$. The resulting equations using this approach require an iterative procedure to be solved and in order to avoid this difficulty, we use the following result:

Lemma 6.1 Let $\mathbf{X}, \mathbf{Y}$ be stable transfer matrices. Then,

$$\|\mathbf{XY}\|_2^2 \leq \|\mathbf{X}\|_2^2 \|\mathbf{Y}\|_\infty^2$$

Proof:

$$\begin{aligned}
\|\mathbf{XY}\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}(\mathbf{XY}(e^{-j\omega})\mathbf{Y}^*\mathbf{X}^*(e^{j\omega})d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{i=0}^{n} \sigma_i^2(\mathbf{XY}(e^{-j\omega}))d\omega \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \bar{\sigma}^2(\mathbf{Y}(e^{-j\omega})) \sum_{i=0}^{n} \sigma_i^2(\mathbf{X}(e^{-j\omega}))d\omega \\
&\leq \sup_{\omega\in[0\ 2\pi]} \bar{\sigma}^2(\mathbf{Y}(e^{-j\omega}))\frac{1}{2\pi}\int_0^{2\pi} \sum_{i=0}^{n} \sigma_i^2(\mathbf{X}(e^{-j\omega}))d\omega \\
&\leq \|\mathbf{X}\|_2^2 \|\mathbf{Y}\|_\infty^2
\end{aligned}$$

■

Using (6.6) and Lemma 6.1,

$$E[\mathrm{tr}(\mathbf{y}\mathbf{y}^T)] \leq \|\mathbf{F}\|_2^2 + \|\mathbf{L}\|_2^2\|\bar{\mathbf{F}}\|_\infty^2 + \cdots \tag{6.7}$$

With this simplification, the decentralized controller that provides an overestimate of the achievable output variance is obtained by solving the following optimization problem

$$\begin{aligned}
&\min_{\mathbf{K}} \|\mathbf{L}\|_2^2 \\
\text{s.t.} \quad &(\mathbf{1}_{nn} - \mathbf{J}) \circ \mathbf{K} = \mathbf{0}
\end{aligned} \tag{6.8}$$

where $\mathbf{1}_{nn}$ is a matrix of ones and $\circ$ is the Hadamard product. $\mathbf{J}$ is a matrix representing the controller structure and is defined as

$$\mathbf{J}_{ij} = \begin{cases} 1 & \text{if } \mathbf{K}_{ij} \neq 0 \\ 0 & \text{if } \mathbf{K}_{ij} = 0 \end{cases} \tag{6.9}$$

6.4 Decentralized MV Benchmark

In this section, an explicit solution to the optimization problem (6.8) is provided. For these purposes, we present the following result, which involves finding the stationary point of a scalar wrt a structured matrix. This result can also be of independent interest.

Lemma 6.2 Let $\mathbf{Y} = \mathbf{X}^T\mathbf{M}\mathbf{X} - \mathbf{N}^T\mathbf{X}$. Then,

$$\frac{\partial[\mathrm{tr}(\mathbf{Y})]}{\partial \mathbf{X}} = \left(\mathbf{M} + \mathbf{M}^T\right)\mathbf{X} - \mathbf{N} \tag{6.10}$$

Proof: Let z_j be the j^{th} column of the Identity matrix. Using the chain rule

$$\begin{aligned}
\frac{\partial[\mathrm{tr}(\mathbf{Y})]}{\partial x_{ij}} &= \mathrm{tr}\left(\frac{\partial \mathbf{X}^T}{\partial x_{ij}}\mathbf{M}\mathbf{X} + (\mathbf{X}^T\mathbf{M} - \mathbf{N}^T)\frac{\partial \mathbf{X}}{\partial x_{ij}}\right) \\
&= \mathrm{tr}\left(z_j z_i^T \mathbf{M}\mathbf{X}\right) + \mathrm{tr}\left((\mathbf{X}^T\mathbf{M} - \mathbf{N}^T) z_i z_j^T\right) \\
&= \mathrm{tr}\left(z_i^T \mathbf{M}\mathbf{X} z_j\right) + \mathrm{tr}\left(z_j^T (\mathbf{X}^T\mathbf{M} - \mathbf{N}^T) z_i\right) \\
&= (\mathbf{M}\mathbf{X})_{ij} + \left(\mathbf{X}^T\mathbf{M} - \mathbf{N}^T\right)_{ji} \\
&= (\mathbf{M}\mathbf{X})_{ij} + \left(\mathbf{M}^T\mathbf{X}\right)_{ij} - \mathbf{N}_{ij}
\end{aligned}$$

Note that (6.10) is a compact representation of the last expression. ■

Proposition 6.1 Let $\mathbf{Y} = \mathbf{X}^T\mathbf{M}\mathbf{X} - \mathbf{N}^T\mathbf{X}$, where $\mathbf{X}$ is a block diagonal matrix. Then, the stationary point of $\mathrm{tr}(\mathbf{Y})$ wrt $\mathbf{X}$ is found by solving

$$\mathbf{J} \circ \left[(\mathbf{M} + \mathbf{M}^T)\right]\mathbf{X} \;=\; \mathbf{J} \circ \mathbf{N} \tag{6.11}$$

where $\mathbf{J}$ is defined similar to (6.9).

Proof: Let $\mathbf{X} = \mathrm{diag}(\mathbf{X}_{11}, \cdots, \mathbf{X}_{MM})$. Then,

$$\mathrm{tr}(\mathbf{Y}) = \sum_{i=1}^{M} \mathrm{tr}(\mathbf{X}_{ii}^T\mathbf{M}_{ii}\mathbf{X}_{ii}) - \mathrm{tr}(\mathbf{N}_{ii})$$

Using lemma 6.2, the stationary point of $\mathrm{tr}(\mathbf{Y})$ wrt $\mathbf{X}_{ii}$ is found by solving

$$\frac{\partial[\mathrm{tr}(\mathbf{Y})]}{\partial \mathbf{X}_{ii}} = \left(\mathbf{M}_{ii} + \mathbf{M}_{ii}^T\right)\mathbf{X}_{ii} - \mathbf{N}_{ii} = \mathbf{0}$$

The result follows by considering the last expression for all i together. ■

6.4.1 Simple Interactor Matrix

If the system has a simple interactor matrix, *i.e.* $\mathbf{D} = q^{-d} \cdot \mathbf{I}$, then $\mathbf{A} = \mathbf{R}\mathbf{G}_w^{-1}$, $\mathbf{B} = \mathbf{F}\mathbf{G}_w^{-1}\tilde{\mathbf{G}}$, where $\mathbf{G}_w = \mathbf{F} + q^{-d}\mathbf{R}$. Using Parseval's equality,

$$\|\mathbf{L}\|_2^2 = \sum_{i=0}^{\infty} \mathrm{tr}(\mathbf{L}_i^T \mathbf{L}_i) \tag{6.12}$$

where $\mathbf{L} = \mathbf{A} - \mathbf{B}\mathbf{K}$ as before and $\mathbf{L}_i$ is the i^{th} impulse response matrix of $\mathbf{L}$ defined as

$$\mathbf{L}_i = \mathbf{A}_i - \sum_{j=0}^{i} \sum_{k=0}^{i-j} \mathbf{B}_j \mathbf{K}_k \tag{6.13}$$

Then, the decentralized MV control law is obtained by finding the stationary point of $\|\mathbf{L}\|_2^2$ wrt $\mathbf{K}_k$, $k = 1, 2 \ldots \infty$ subject to the structural constraint on the controller. For numerical reasons, however, it is necessary to approximate $\mathbf{A}, \mathbf{B}$ and $\mathbf{K}$ by finite impulse response models having order N. Using Lemma 6.1, the stationary point is found by solving,

$$\frac{\partial \|\mathbf{L}\|_2^2}{\partial \mathbf{K}_k} = \mathbf{J} \circ \left[\sum_{i=0}^{N-k} \mathbf{B}_i^T \mathbf{L}_{i+k} \right] = \mathbf{0} \tag{6.14}$$

To simplify notation in the further treatment, we define the following linear operator,

Definition 6.3 Let $\mathbf{X}, \mathbf{Y}$ be defined such that $\dim(\mathbf{X}) = \dim(\mathbf{Y}_{ij})$ for all i, j. Then, the *block-wise Kronecker-Hadamard product* is defined as,

$$\mathbf{X} \oslash \mathbf{Y} = \begin{bmatrix} \mathbf{X} \circ \mathbf{Y}_{11} & \mathbf{X} \circ \mathbf{Y}_{12} & \cdots \\ \mathbf{X} \circ \mathbf{Y}_{21} & \mathbf{X} \circ \mathbf{Y}_{22} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

A rearrangement of (6.14) gives,

$$\left[\mathbf{J} \oslash (\mathbf{B}_H^T \mathbf{B}_H) \right] \mathbf{K}_C = \mathbf{J} \oslash (\mathbf{B}_H^T \mathbf{A}_C) \tag{6.15}$$

where $\mathbf{A}_C$ and $\mathbf{K}_C$ contain the impulse response matrices of $\mathbf{A}$ and $\mathbf{K}$ respectively, and $\mathbf{B}_H$ is a lower block triangular Hankel matrix. The $\mathbf{A}_C$, $\mathbf{K}_C$ and $\mathbf{B}_H$ are defined as

$$\begin{aligned} \mathbf{K}_C &= \begin{bmatrix} \mathbf{K}_0^T & \mathbf{K}_1^T & \mathbf{K}_2^T & \cdots \mathbf{K}_N^T \end{bmatrix}^T \\ \mathbf{A}_C &= \begin{bmatrix} \mathbf{A}_0^T & \mathbf{A}_1^T & \mathbf{A}_2^T & \cdots \mathbf{K}_N^T \end{bmatrix}^T \\ \mathbf{B}_H &= \begin{bmatrix} \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{B}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{B}_N & \mathbf{B}_{N-1} & \cdots & \cdots & \mathbf{B}_0 \end{bmatrix} \end{aligned} \tag{6.16}$$

When $\left[\mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{B}_H\right)\right]$ is invertible, the suboptimal decentralized MV controller is given as,

$$\mathbf{K}_C = \left[\mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{B}_H\right)\right]^{-1} \left[\mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{A}_C\right)\right] \tag{6.17}$$

Remark 6.1 Since $\mathbf{J}$ always has full rank, rank deficiency of $\mathbf{B}_H^T \mathbf{B}_H$ makes $\left[\mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{B}_H\right)\right]$ singular. This happens when some of $\mathbf{B}_i$'s are singular. For a system with simple interactor matrix, $\mathbf{B} = \mathbf{F}\mathbf{G}_w^{-1}\tilde{\mathbf{G}}$ has no infinite zeros and thus $\mathbf{B}_i$ is nonsingular for all i.

The earlier developments in this section are summarized by the following result:

Proposition 6.2 Consider the system (6.2) with a simple interactor matrix. Define $\mathbf{A} = \mathbf{R}\mathbf{G}_w^{-1}$, $\mathbf{B} = \mathbf{F}\mathbf{G}_w^{-1}\tilde{\mathbf{G}}$. Then, a suboptimal solution to finding a decentralized controller that minimizes $E[\mathrm{tr}(\mathbf{y}\mathbf{y}^T)]$ is given by (6.17).

Let $\mathbf{y}_{mvd}$ be the output of the closed loop system under the optimal decentralized MV control law. Then, a decentralized performance index is defined as

$$\eta_{mvd} = \frac{\mathrm{E}[\mathrm{tr}(\mathbf{y}_{mvd}\mathbf{y}_{mvd}^T)]}{\mathrm{E}[\mathrm{tr}(\mathbf{y}\mathbf{y}^T)]} \tag{6.18}$$

The full multivariable performance index η_{mv} is defined similarly, where $\eta_{mv} \leq \eta_{mvd}$. Ideally, $0 \leq \eta_{mvd} \leq 1$, but when evaluated based on the suboptimal decentralized controller given by (6.17), η_{mvd} may exceed 1. In any case, a value of η_{mvd} close to zero always indicates poor performance.

In certain special cases, the decentralized controller given by (6.17) is optimal. For example, when $\mathbf{J} = \mathbf{1}_{nn}$, (6.17) reduces to the optimal full multivariable MV control law. Similarly, when $\mathbf{N} = \mathbf{I}$ or the system is affected by white noise, $\mathbf{K}_C = \mathbf{0}$, which is optimal.

Remark 6.2 When $\mathbf{F}$ commutes with $\mathbf{K}$, use of Lemma 6.1 to simplify (6.6) to (6.7) is not required. In this case, better estimates of η_{mvd} are obtained by redefining $\mathbf{A} = \mathbf{R}\mathbf{G}_w^{-1}\mathbf{F}$, $\mathbf{B} = \mathbf{F}\mathbf{G}_w^{-1}\tilde{\mathbf{G}}\mathbf{F}$ and using Proposition 6.2 as before.

Example 6.2 We revisit example 6.1. The variation of η_{mv} and η_{mvd} with k is shown in Figure 6.2. For $k = 0.17$, $\eta_{mvd} \approx 0.82$, which is large compared to $\eta_{mv} \approx 0.6$. This justifies our earlier remark that the decentralized structure puts an inherent limitation on the achievable performance for this system and no significant performance improvement is possible by controller re-tuning.

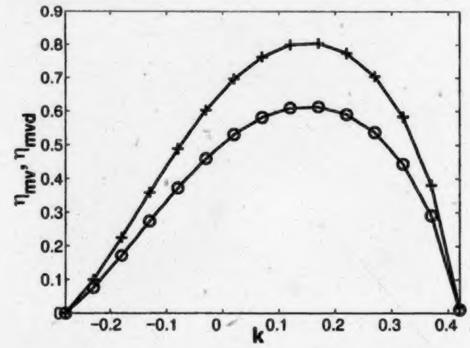

Figure 6.2: Comparison of η_{mv} (o) and η_{mvd} (+) for Example 6.2. The controller structure limits the achievable output performance.

Proposition 6.2 can also be used for input-output pairing selection. For this system, the upper bound on achievable output performance for pairing on the diagonal and off-diagonal elements is 18.99 and 16.02 respectively. Based on this criterion, the latter alternative may be preferred.

6.4.2 General Interactor Matrix

When the system has a general interactor matrix, $\mathbf{B}$ is non-invertible due to presence of infinite zeros (see Remark 6.1) and some modifications are required. Let $\mathbf{D}_B$ be the unitary interaction matrix of $\mathbf{B}$ and $\tilde{\mathbf{B}} = \mathbf{D}_B\mathbf{B}$. Then

$$\begin{aligned}\|\mathbf{L}\|_2^2 &= \|\mathbf{A} - \mathbf{D}_B^{-1}\tilde{\mathbf{B}}\mathbf{K}\|_2^2 \\ &= \|\mathbf{D}_B\mathbf{A} - \tilde{\mathbf{B}}\mathbf{K}\|_2^2 = \|\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{K}\|_2^2\end{aligned}$$

The suboptimal decentralized controller is obtained by following the same steps as before:

$$\mathbf{K}_C = \left[\mathbf{J} \oslash \left(\tilde{\mathbf{B}}_H^T\tilde{\mathbf{B}}_H\right)\right]^{-1}\left[\mathbf{J} \oslash \left(\tilde{\mathbf{B}}_H^T\tilde{\mathbf{A}}_C\right)\right] \tag{6.19}$$

where $\tilde{\mathbf{A}}_C, \tilde{\mathbf{B}}_H$ are defined similar to (6.16).

Proposition 6.3 Consider the system (6.2) with a general interactor matrix. Define $\tilde{\mathbf{A}} = \mathbf{D}_B\bar{\mathbf{R}}\mathbf{G}_w^{-1}$, $\tilde{\mathbf{B}} = \mathbf{D}_B\bar{\mathbf{F}}\bar{\mathbf{G}}_d^{-1}\tilde{\mathbf{G}}$, where $\mathbf{D}_B$ is the unitary interactor matrix of $\bar{\mathbf{F}}\bar{\mathbf{G}}_d^{-1}\tilde{\mathbf{G}}$. Then, a suboptimal solution to finding a decentralized controller that minimizes $E[\mathrm{tr}(\mathbf{y}\mathbf{y}^T)]$ is given by (6.19).

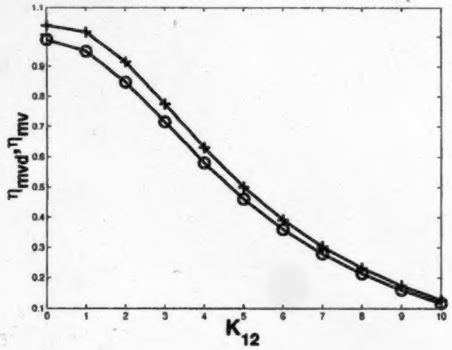

Figure 6.3: Comparison of η_{mv} (o) and η_{mvd} (+) for Example 6.3. The controller structure poses no serious limitations.

In the previous example, controller structure posed significant limitations on the achievable performance. This is not always the case, as shown below:

Example 6.3 Consider the following system adapted from Huang and Shah [69],

$$\mathbf{G} = \begin{bmatrix} \frac{q^{-1}}{1-0.4q^{-1}} & \frac{K_{12}q^{-2}}{1-0.1q^{-1}} \\ \frac{0.3q^{-1}}{1-0.1q^{-1}} & \frac{q^{-2}}{1-0.8q^{-1}} \end{bmatrix} \quad \mathbf{G}_w = \begin{bmatrix} \frac{1}{1-0.5q^{-1}} & \frac{-0.6}{1-0.5q^{-1}} \\ \frac{0.5}{1-0.5q^{-1}} & \frac{1}{1-0.5q^{-1}} \end{bmatrix}$$

where the variable K_{12} controls the extent of interaction among the variables. The objective is to compare the performance of the following controller for different values of K_{12}.

$$\mathbf{K} = \begin{bmatrix} \frac{0.5-0.2q^{-1}}{1-0.5q^{-1}} & 0 \\ 0 & \frac{0.25-0.2q^{-1}}{(1-0.5q^{-1})(1+0.5q^{-1})} \end{bmatrix}$$

The η_{mvd}, η_{mv} for various K_{12} are shown in Figure 6.3. For each value of K_{12}, there exists a decentralized controller that closely matches the performance of the optimal full multivariable controller. Hence, the controller structure poses no serious limitation on the achievable performance for this system. This further illustrates that large interactions do not necessarily limit the performance of decentralized controllers compared to the full multivariable controllers.

6.5 Achievable PID Performance

The suboptimal decentralized controller is expressed in terms of its impulse response matrices. By restricting the order of the controller or setting $\mathbf{K}_k = 0$ for all $k > p$, controllers with reduced complexity can be obtained. In this section, this approach is used to find an overestimate on achievable output variance using multi-loop PID controllers, which are expressed as,

$$\mathbf{K}_{\text{PID}} = \frac{1}{\Delta}\sum_{i=0}^{2}\mathbf{C}_i q^{-i} = \frac{1}{\Delta}\mathbf{C}$$

where $\Delta = 1 - q^{-1}$. By considering $1/\Delta$ as a part of $\tilde{\mathbf{G}}$ and minimizing $\|\mathbf{L}\|_2^2$ wrt $\mathbf{C}$, an overestimate of the achievable PID performance can be derived. Then Propositions 6.2 and 6.3 can be used by limiting the column dimensions of $\mathbf{A_C}, \mathbf{B_H}$ to $3n$. To ensure that the assumption of stability of $\mathbf{G}$ is satisfied, the integrator can be moved just inside the unit circle without affecting the result significantly. In general, controllers with reduced complexity having order p can be obtained by limiting the column dimensions of $\mathbf{A_C}, \mathbf{B_H}$ to pn.

Example 6.4 Consider the following system taken from Ko and Edgar [75],

$$y = \frac{q^{-6}}{1-0.8q^{-1}}u + \frac{1-0.2q^{-1}}{(1-0.3q^{-1})(1+0.4q^{-1})(1-0.5q^{-1})}a$$

Clearly the results presented earlier also hold for SISO systems. Based on these results, the achievable output variances under MV and PI control are 1.11 showing that the control structure poses no limitations. However, when the disturbance model contains an additional integrator, the achievable output variances under MV and PI control are 11.95 and 17.86 respectively. The achievable performances differ by more than 50% revealing the effect of controller structure on achievable performance. Note that for both these cases, the achievable PI performance is close to the results obtained by Ko and Edgar [75], who used numerical search.

6.6 Limitations

The results presented in this paper require that the system's model be fully known. This can be very demanding for online performance monitoring of industrial systems, especially in presence of changing operating conditions. The requirement of knowledge of the system's model can be partially relaxed by estimating $\mathbf{G}_w$ using regular operating data, as suggested by Ko and Edgar [75]. Example 6.3 shows that the controller structure does not always limit the achievable performance. The identification of $\mathbf{G}$ should only be undertaken if large differences are seen between the actual output variance and MV benchmark for full multivariable controllers.

The suboptimal controller is expressed in terms of its impulse response matrices, whose determination is computationally inexpensive. Starting from a low value, the controller order can be gradually increased until convergence, but convergence can be extremely slow in some cases. This difficulty is overcome by recognizing that $\left[\mathbf{J} \oslash \left(\mathbf{B}_H^T \mathbf{B}_H\right)\right]$ is a sparse Toeplitz matrix and using available computationally efficient methods (*e.g.*, Brent *et al.* [16]) for its inversion.

The decentralized MV control law is based on an approximation of the closed loop expression and thus stability is not guaranteed. A possible approach to overcome this limitation is to reduce the gain of the decentralized controller until stability is achieved, however, such an approach increases the sub-optimality of the results.

6.7 Chapter Summary

For performance assessment purposes, ignoring the controller structure can lead to incorrect conclusions regarding significant performance improvement through controller tuning. In this chapter, we presented an approximate solution to the decentralized minimum variance control problem, which provides an overestimate of the achievable output variance without numerical search. The proposed method can easily handle the case of multi-loop PID controllers. The primary limitation of the proposed method is that complete knowledge of the system's model is required and some recommendations are provided to partially overcome this limitation.

Chapter 7

Conclusions and Future Work

7.1 Thesis Conclusions

In this thesis, we developed tools for handling different aspects of the control configuration design (CCD) problem. The major contributions are listed below:

- The achievable input performance is characterized for FDLTI systems possibly having time delay in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control frameworks.
- A method for finding a stabilizing decentralized controller through independent designs is presented. This method extends the practical applicability of the μ-interaction measure to unstable systems.
- The problem of finding an optimal block diagonal approximation of a multivariate system is introduced and a numerical solution is proposed.
- Many new algebraic properties of block relative gain (BRG) are developed. The connection between BRG and important closed loop properties is explored and some common conjectures are corrected.
- The problem of establishing existence of diagonal controller such that the system has integrity against controller failure is shown to be NP-hard.
- A suboptimal, yet explicit solution to the decentralized minimum variance benchmark problem is proposed.

In many cases, the CCD problem can be reasonably solved using the tools presented in this thesis alone or with possible minor extensions. For example, reliable decentralized controller can be designed for open loop stable systems using the results of Chapters 4- 5.

In the context of the CCD problem, the results on Decentralized MV benchmark are useful for screening of pairing alternatives with achievable output performance as a criterion.

We have not handled the important issue of model uncertainty explicitly. Note that minimization of input energy required for stabilization provides the maximally robust controller for norm bounded additive uncertainty [43]. The results of Chapter 3 can also be easily extended for handling robust stability and performance issues using the approach available in [101]. It must be acknowledged; however, that solving the CCD or control structure design problem for general time-varying non-linear systems remains an open challenge and this thesis can be seen as a positive step in that direction.

7.2 Directions for Future Work

We pointed out some potential directions for generalizing and improving upon the results presented in this thesis in the summaries of the individual chapters. Some other relevant issues are discussed below with the hope that solving these problems will move us closer towards finding a general solution for the CSD problem.

- The characterization of achievable performance has received increasing interest from researchers, but the effect of controller structure on the achievable performance remains unclear. The results of Zames and Bensoussan [116] can be seen as a good starting point in this direction.

- It is likely that the optimal solution to the block diagonal approximation problem is not unique. An analytical solution is necessary to characterize all possible solutions. To this end, it is useful to approach the $\mathcal{H}_\infty$ optimal block diagonal approximation problem for stable systems using the results of Glover *et al.* [44].

- In some cases, it may not be possible to find a stabilizing decentralized controller through independent designs. This difficulty can be partially overcome by extending the ideas presented in Chapter 3 for sequential design of decentralized controller for unstable systems.

- The requirement that a model be available hinders the online implementation of the decentralized minimum variance benchmark. It would be extremely useful, if exact or approximate methods can be derived, where this stringent requirement can be relaxed.

- There are no practical tools available (other than numerical simulation) for directly handling the non-linear behavior of the process systems. An indirect approach is to approximate the system as a nominal model with an associated uncertainty description [11], but the involved computational complexity is limiting.

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